Coupled and Uncoupled Sign-changing Spikes of Singularly Perturbed Elliptic Systems

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Outline

Introduction

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- We study the **existence and asymptotic behavior** of *semi-positive* solutions, i. e. having **positive and sign-changing components** to the singularly perturbed system of elliptic equations

$$\begin{cases} -\varepsilon^2 \Delta u_i + u_i = \mu_i |u_i|^{p-2} u_i + \sum\limits_{\substack{j=1 \\ j \neq i}}^{\ell} \lambda_{ij} \beta_{ij} |u_j|^{\alpha_{ij}} |u_i|^{\beta_{ij}-2} u_i, \\ u_i \in H_0^1(\Omega), \quad u_i \neq 0, \qquad i = 1, \dots, \ell, \end{cases} \tag{$\mathcal{S}_{\Omega,\varepsilon}$}$$

- $\varepsilon > 0$ is a small parameter, Ω is a bounded smooth domain in $\mathbb{R}^N$,
- $\mathbf{N \geq 4}, \quad \mu_\mathbf{i} > \mathbf{0}, \quad \lambda_\mathbf{ij} = \lambda_\mathbf{ji} < \mathbf{0}, \quad \alpha_\mathbf{ij}, \beta_\mathbf{ij} > \mathbf{1}, \quad \alpha_\mathbf{ij} = \beta_\mathbf{ji},$

$$\alpha_\mathbf{ij} + \beta_\mathbf{ij} = \mathbf{p} \in (\mathbf{2}, \mathbf{2}^*),$$

and $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent.

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  (2): the limit profile is **a solution of the uncoupled system**, i.e., after rescaling and translation, the limit profile of the i-th component is a positive or a nonradial sign-changing solution to the equation

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- This system arises as a model for various physical phenomena, in particular in the study of standing waves for **a mixture of Bose-Einstein condensates** of ℓ different hyperfine states which overlap in space, see for example:

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- We consider the case in which the interaction between particles in **the same state is attractive ($\mu_i > 0$)** and the interaction between particles in **any two different states is repulsive ($\lambda_{ij} < 0$)**.

Main References

- **Lin and Wei (2005)** (for $N = 2, 3$)

  - Described the behavior of **positive least energy solutions** for the system $(\mathcal{S}_{\Omega,\varepsilon})$ with cubic nonlinearity ($\alpha_{ij} = \beta_{ij} = 2$) as $\varepsilon \to 0$;

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- **Bartsch and Willem (1993)** (for $N = 4$ and $N \geq 6$)
- **Lorca and Ubilla (2004)** (for $N = 5$)
- **Clapp and Srikanth (2016)** (for $N \geq 4$)

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(3) - Bartsch, Thomas; Willem, Michel: Infinitely many nonradial solutions of a Euclidean scalar field equation. J. Funct. Anal. 117 (1993), no. 2, 447-460.

(4) - Lorca, Sebastián; Ubilla, Pedro: Symmetric and nonsymmetric solutions for an elliptic equation on $\mathbb{R}^N$. Nonlinear Anal. 58 (2004), no. 7-8, 961-968.

(5) - Clapp, Mónica; Srikanth, P. N.: Entire nodal solutions of a semilinear elliptic equation and their effect on concentration phenomena. J. Math. Anal. Appl. 437 (2016), no. 1, 485-497.

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(7) - Sirakov, Boyan: Least energy solitary waves for a system of nonlinear Schrödinger equations in Rn. Comm. Math. Phys. 271 (2007), no. 1, 199-221.

(8) - Sato, Yohei; Wang, Zhi-Qiang: On the least energy sign-changing solutions for a nonlinear elliptic system. Discrete Contin. Dyn. Syst. 35 (2015), no. 5, 2151-2164.

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Let $\mathbf{N = 4}$ **or** $\mathbf{N \geq 6}$. *For any given* $0 \leq m \leq \ell$, *the system* $(\mathcal{S}_{\infty,\ell})$ *has a solution* $\mathbf{w} = (w_1, \ldots, w_\ell)$ *whose first* m *components* $w_1, \ldots, w_m$ **are positive** *and whose last* $\ell - m$ *components* $w_{m+1}, \ldots, w_\ell$ **are nonradial and change sign**. *Furthermore,* $\mathbf{w}$ *satisfies*

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$$\begin{cases} w_i(z_1, z_2, x) = w_i(\mathrm{e}^{\mathrm{i}\vartheta} z_1, \mathrm{e}^{\mathrm{i}\vartheta} z_2, gx) & \text{for all } \vartheta \in [0, 2\pi),\ g \in O(N-4),\ i = 1, \ldots, \ell, \\ w_i(z_1, z_2, x) = w_i(z_2, z_1, x) & \text{if } i = 1, \ldots, m, \\ w_i(z_1, z_2, x) = -w_i(z_2, z_1, x) & \text{if } i = m+1, \ldots, \ell, \end{cases} \tag{1.1}$$

for all $(z_1, z_2, x) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4} \equiv \mathbb{R}^N$, *and it has a* **least energy** *among all nontrivial solutions with these symmetry properties.*

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Let $\ell \geq 2$. System $(\mathcal{S}_{\infty,\ell})$ has a least energy solution satisfying (1.1) **if and only if** $\mathrm{Fix}(G) := \{x \in \mathbb{R}^N : gx = x \ \forall \ g \in G\} = \{0\}$.

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- For $\mathbf{N = 5}$ **there is no** such a subgroup G of symmetries satisfying $\mathrm{Fix}(G) = \{0\}$.

About the singly perturbed system $(\mathcal{S}_{\Omega,\varepsilon})$

- Set $\|u\|_\varepsilon^2 := \dfrac{1}{\varepsilon^N}\displaystyle\int_{\mathbb{R}^N}\left[\varepsilon^2|\nabla u|^2 + u^2\right]$ and $\|u\| := \|u\|_1$ for $\varepsilon > 0$ and $u \in H^1(\mathbb{R}^N)$, we obtain **two different** asymptotic behaviors as $\varepsilon \to 0$:

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Theorem (3) - (Coupled Spikes Solutions)

Let $\mathbf{N = 4}$ **or** $\mathbf{N \geq 6}$*, and* $\Omega = B_1(0)$*. Then, for any given* $0 \leq m \leq \ell$ *and any sequence* (ε_k) *of positive numbers converging to zero, there exists solution* $\widehat{\mathbf{u}}_k = (\widehat{u}_{1k}, \ldots, \widehat{u}_{\ell k})$ *to the system* $(\mathcal{S}_{\Omega,\varepsilon_k})$ *whose* **first** m **components are positive** *and whose* **last** $\ell - m$ **components are nonradial and change sign***, with the following limit profile:*

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The first m *components of* $\mathbf{w}$ *are positive, its last* $\ell - m$ *components are nonradial and change sign, and* $\mathbf{w}$ *satisfies* (1.1). *Therefore,*

$$\lim_{k\to\infty}\sum_{i=1}^{\ell}\|\widehat{u}_{ik}\|_{\varepsilon_k}^2 = \sum_{i=1}^{\ell}\|w_i\|^2 =: \widehat{\mathfrak{c}}_m.$$

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Let $\mathbf{N} \geq \mathbf{5}$ *and* $\Omega = B_1(0)$*. Then, for any given* $0 \leq m \leq \ell$ *and any sequence* (ε_k) *of positive numbers converging to zero, there exists solution* $\mathbf{u}_k = (u_{1k}, \dots, u_{\ell k})$ *to the system* $(\mathcal{S}_{\Omega,\varepsilon_k})$ *whose* **first** m **components are positive** *and whose* **last** $\ell - m$ **components are nonradial and change sign**, *with the following limit profile:*

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$v_1, \ldots, v_m$ *are* **positive and radial functions***, while* $v_{m+1}, \ldots, v_\ell$ *are sign-changing, nonradial functions and for* $(z_1, z_2, x) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4}$ *they satisfy*

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Furthermore, $\lim_{k\to\infty} \sum_{i=1}^{\ell} \|u_{ik}\|_{\varepsilon_k}^2 = \sum_{i=1}^{\ell} \|v_i\|^2 =: \mathfrak{c}_m$, *satisfies* $\mathfrak{c}_{\mathbf{m}} < \widehat{\mathfrak{c}}_{\mathbf{m}}$**,** *with* $\widehat{\mathfrak{c}}_m$ *as in Theorem 3, if* $\mathbf{N} \geq \mathbf{6}$.

Behavior of Solutions

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  They **concentrate at points that are far from each other** and from the boundary of the ball;

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- The **sign of the interaction coefficient** λ_{ij} **is not determinant in the segregation behavior** of higher energy solutions;
- Since they have **higher energy**, they enjoy **more symmetries** than those given by Theorem 4.

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  - If the fixed-point space has **positive dimension**, all components will **move far away** from each other;
- For **$\mathbf{N = 3}$ there are no symmetries** with the properties required to produce nonradial sign-changing solutions;

Theoretical Background

A Suitable Subgroup of Symmetries

- Let G be a **closed subgroup of $O(N)$**, denote by $\mathbf{Gx} := \{\mathbf{gx} : \mathbf{g} \in \mathbf{G}\}$ the G-orbit of $x \in \mathbb{R}^N$;

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- If ϕ is **surjective**, then every nontrivial function $u \in H^1_0(\Theta)^\phi$ is **nonradial and changes sign**.

Relevant Examples

(i) Let Γ be the group generated by $\{\mathrm{e}^{\mathrm{i}\vartheta} : \vartheta \in [0, 2\pi)\} \cup \{\tau\}$ acting on $\mathbb{R}^N$ by

$$\mathrm{e}^{\mathrm{i}\vartheta}(z_1, z_2, x) = (\mathrm{e}^{\mathrm{i}\vartheta} z_1, \mathrm{e}^{\mathrm{i}\vartheta} z_2, x) \quad \text{and} \quad \tau(z_1, z_2, x) = (z_2, z_1, x)$$

for all $(z_1, z_2, x) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4} \equiv \mathbb{R}^N$ and $\phi : \Gamma \to \mathbb{Z}_2$ be the homomorphism given by $\phi(\mathrm{e}^{\mathrm{i}\vartheta}) := 1$ and $\phi(\tau) := -1$.

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$$K^\phi x_0 = \{(\mathrm{e}^{\mathrm{i}\vartheta}, 0, 0) : \vartheta \in [0, 2\pi)\} \quad \text{and}$$

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so $(\mathbf{A_1})$ is satisfied.

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with $\varepsilon > 0$, $\mu_i > 0$, $\lambda_{ij} = \lambda_{ji} < 0$, $\alpha_{ij}, \beta_{ij} > 1$, $\alpha_{ij} = \beta_{ji}$ and $\alpha_{ij} + \beta_{ij} = p \in (2, 2^*)$.

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- Set $\mathcal{H}^\ell(\Theta) := H_0^1(\Theta)^{\phi_1} \times \cdots \times H_0^1(\Theta)^{\phi_\ell}$, and denote an element in $\mathcal{H}^\ell(\Theta)$ by $\mathbf{u} = (u_1, \dots, u_\ell)$. For each $\varepsilon > 0$ we define the equivalent norms

$$\|\mathbf{u}\|_{\ell,\varepsilon} := \left(\sum_{i=1}^{\ell} \|u_i\|_\varepsilon^2 \right)^{1/2}, \text{where } \|u_i\|_\varepsilon^2 := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \left[\varepsilon^2 |\nabla u_i|^2 + u_i^2 \right].$$

Variational Setting

- Consider the functional $\mathcal{J}_\varepsilon^\ell : \mathcal{H}^\ell(\Theta) \to \mathbb{R}$ given by

$$\mathcal{J}_\varepsilon^\ell(\mathbf{u}) := \frac{1}{2}\sum_{i=1}^{\ell} \|u_i\|_\varepsilon^2 - \frac{1}{p}\sum_{i=1}^{\ell} \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \mu_i |u_i|^p - \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{\ell} \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \lambda_{ij}|u_j|^{\alpha_{ij}}|u_i|^{\beta_{ij}},$$

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- Since $\lambda_{ij} = \lambda_{ji}$, $\beta_{ij} = \alpha_{ji}$ and $\alpha_{ij} + \beta_{ij} = p$, its partial derivatives are

$$\begin{aligned} \partial_i \mathcal{J}_\varepsilon^\ell(\mathbf{u})v = \frac{1}{\varepsilon^N}\Big[&\int_{\mathbb{R}^N} (\varepsilon^2 \nabla u_i \cdot \nabla v + u_i v) \\ &- \int_{\mathbb{R}^N} \mu_i |u_i|^{p-2} u_i v - \sum_{\substack{j=1\\ j\neq i}}^{\ell} \int_{\mathbb{R}^N} \lambda_{ij}\beta_{ij}|u_j|^{\alpha_{ij}}|u_i|^{\beta_{ij}-2}u_i v\Big], \end{aligned} \tag{2.1}$$

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  for $v \in H_0^1(\Theta)^{\phi_i}$ and $i = 1, \dots, \ell$.

- By the **principle of symmetric criticality**, the solutions to system $(\mathcal{S}_{\Theta,\varepsilon}^\phi)$ are the critical points of $\mathcal{J}_\varepsilon^\ell$ whose components u_i are nontrivial.

The Nehari-Type Set

- All solutions belong to the Nehari-type set

$$\mathcal{N}_\varepsilon^\ell(\Theta) := \Big\{\mathbf{u} \in \mathcal{H}^\ell(\Theta) : u_i \neq 0,\ \partial_i \mathcal{J}_\varepsilon^\ell(\mathbf{u}) u_i = 0,\ \forall\ i = 1, \ldots, \ell\Big\}.$$

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- Note that

$$\partial_i \mathcal{J}_\varepsilon^\ell(\mathbf{u})u_i = \|u_i\|_\varepsilon^2 - \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \mu_i |u_i|^p - \sum_{\substack{j=1\\ j\neq i}}^{\ell} \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \lambda_{ij}\beta_{ij}|u_j|^{\alpha_{ij}}|u_i|^{\beta_{ij}}. \tag{2.2}$$

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- Define

$$c_\varepsilon^\ell(\Theta) := \inf_{\mathbf{u}\in\mathcal{N}_\varepsilon^\ell(\boldsymbol{\Theta})} \mathcal{J}_\varepsilon^\ell(\mathbf{u}).$$

- From (2.1), if $\mathbf{u} = (u_1, u_2, \ldots, u_\ell) \in \mathcal{N}_\varepsilon^\ell(\Theta)$ one sees that

$$\mathcal{J}_\varepsilon^\ell(\mathbf{u}) = \frac{p-2}{2p}\sum_{i=1}^{\ell} \|u_i\|_\varepsilon^2 = \frac{p-2}{2p}\|\mathbf{u}\|_{\ell,\varepsilon}^{\mathbf{2}}. \tag{2.3}$$

The Existence of a Least Energy Solution

- A solution to the system $(\mathcal{S}^{\phi}_{\Theta,\varepsilon})$ such that $\mathcal{J}^{\ell}_{\varepsilon}(\mathbf{u}) = \mathbf{c}^{\ell}_{\varepsilon}(\boldsymbol{\Theta})$ is called a **least energy solution** to $(\mathcal{S}^{\phi}_{\Theta,\varepsilon})$.

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Theorem (5) - (Existence of minimizers)

If Θ is a bounded G-invariant domain in $\mathbb{R}^N$, then, for each $\varepsilon > 0$, system $(\mathcal{S}^{\phi}_{\Theta,\varepsilon})$ has a **least energy solution***.*

The Limit System

Establishing the Nehari Levels

- Let ϕ_i and G satisfy $(\mathbf{A_1})$ and also the following property:

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$$\begin{cases} -\Delta u_i + u_i = \mu_i |u_i|^{p-2} u_i + \sum\limits_{\substack{j=1 \\ j \neq i}}^{\ell} \lambda_{ij} \beta_{ij} |u_j|^{\alpha_{ij}} |u_i|^{\beta_{ij}-2} u_i, \\ u_i \in H^1(\mathbb{R}^N)^{\phi_i}, \quad u_i \neq 0, \qquad 1 < i \leq \ell. \end{cases} \qquad (\mathcal{S}^{\phi}_{\infty,\ell})$$

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- Set $\mathcal{H}^\ell := H^1(\mathbb{R}^N)^{\phi_1} \times \cdots \times H^1(\mathbb{R}^N)^{\phi_\ell}$, $\mathcal{J}^\ell_\infty := \mathcal{J}^\ell_1 : \mathcal{H}^\ell \to \mathbb{R}$ and $\mathcal{N}^\ell_\infty := \mathcal{N}^\ell_1(\mathbb{R}^N)$ and also let $c^\ell_\infty := \inf\limits_{\mathbf{u} \in \mathcal{N}^\ell_\infty} \mathcal{J}^\ell_\infty(\mathbf{u})$.

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- For each $i = 1, \dots, \ell$, consider the problem

$$\begin{cases} -\Delta u + u = \mu_i |u|^{p-2} u, \\ u \in H^1(\mathbb{R}^N)^{\phi_i}, \quad u \neq 0. \end{cases} \qquad (\mathcal{P}^{\phi_i}_i)$$

Establishing the Nehari Levels

- Its solutions are the critical points of $J_i : H^1(\mathbb{R}^N)^{\phi_i} \to \mathbb{R}$ given by $J_i(u) := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2) - \frac{1}{p}\int_{\mathbb{R}^N}|u|^p$, and belong to the Nehari manifold $\mathcal{N}_i := \{u \in H^1(\mathbb{R}^N)^{\phi_i} : u \neq 0,\ J_i'(u)u = 0\}$. We set

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- A solution $\mathbf{u}$ to $(\mathcal{S}^{\phi}_{\infty,\ell})$ satisfying $\mathcal{J}^{\ell}_{\infty}(\mathbf{u}) = \mathbf{c}^{\ell}_{\infty}$ is called a **least energy solution** to $(\mathcal{S}^{\phi}_{\infty,\ell})$. Similarly, a solution u to $(\mathcal{P}^{\phi_i}_i)$ satisfying $J_i(u) = c_i$ is called a **least energy solution** to $(\mathcal{P}^{\phi_i}_i)$.

Establishing the Nehari Levels

- Its solutions are the critical points of $J_i : H^1(\mathbb{R}^N)^{\phi_i} \to \mathbb{R}$ given by $J_i(u) := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2) - \dfrac{1}{p}\int_{\mathbb{R}^N}|u|^p$, and belong to the Nehari manifold $\mathcal{N}_i := \{u \in H^1(\mathbb{R}^N)^{\phi_i} : u \neq 0,\ J_i'(u)u = 0\}$. We set

$$c_i := \inf_{u \in \mathcal{N}_i} J_i(u).$$

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- Concerning to the fundamental role of the *G-fixed-point space*

$$\mathrm{Fix}(G) := \{x \in \mathbb{R}^N : gx = x \text{ for all } g \in G\}$$

we prove the following results.

Proposition (6) - (Comparing the Nehari Levels)

The following statements hold true:

(i) $c_\infty^\ell \geq \sum_{i=1}^{\ell} c_i$.

(ii) *If* $\mathrm{Fix}(G)$ *has* **positive dimension**, *then* $c_\infty^\ell = \sum_{i=1}^{\ell} c_i$.

(iii) *If* $\ell \geq 2$ *and* $c_\infty^\ell = \sum_{i=1}^{\ell} c_i$, *then* c_∞^ℓ *is* **NOT attained**.

Theorem (7) - (The Minimizing Sequences of System $(\mathcal{S}^{\phi}_{\infty,\ell})$)

Let $\ell \geq 2$ and $\mathbf{u}_k = (u_{1k}, \ldots, u_{\ell k}) \in \mathcal{N}^{\ell}_{\infty}$ be such that $\mathcal{J}^{\ell}_{\infty}(\mathbf{u_k}) \to \mathbf{c}^{\ell}_{\infty}$.

(*I*) *If $\mathrm{Fix}(\mathbf{G}) = \{\mathbf{0}\}$, then there exists a* **least energy solution** *$\mathbf{w} = (w_1, \ldots, w_\ell)$ to the system $(\mathcal{S}^{\phi}_{\infty,\ell})$ such that, after passing to a subsequence, $\lim_{k\to\infty} \|u_{ik} - w_i\| = 0$ for all $i = 1, \ldots, \ell$, and*

$$c^{\ell}_{\infty} > \sum_{i=1}^{\ell} c_i.$$

Moreover, if $u_{ik} \geq 0$ for all $k \in \mathbb{N}$, then $w_i \geq 0$.

Behavior of the Minimizing Sequence of System $(\mathcal{S}^{\phi}_{\infty,\ell})$

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(*II*) *If $\mathbf{dim}(\mathrm{Fix}(G)) > 0$, then, for each $i = 1, \ldots, \ell$, there exist (ξ_{ik}) in $\mathrm{Fix}(G)$ and a* **least energy solution** *v_i to the problem $(\mathcal{P}^{\phi_i}_i)$ such that, after passing to a subsequence, $\lim_{k\to\infty} |\xi_{ik} - \xi_{jk}| = \infty$ if $i \neq j$, $\lim_{k\to\infty} \|u_{ik} - v_i(\,\cdot\, - \xi_{ik})\| = 0$ for all $i, j = 1, \ldots, \ell$, and $\mathbf{c}^{\ell}_{\infty} = \sum_{\mathbf{i=1}}^{\ell} \mathbf{c_i}$. Moreover, if $u_{ik} \geq 0$ for all $k \in \mathbb{N}$, then $v_i \geq 0$.*

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▷ The group $\mathbf{G} := \mathbf{\Gamma} \times \mathbf{O(N-4)}$ introduced in Example (*ii*) satisfies $(\mathbf{A_2})$ if either $N = 4$ **or** $N \geq 6$**, and** $\mathrm{Fix}(G) = \{0\}$;

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$\square$

The Asymptotic Behavior of Minimizers

The Limit Profile of the Least Energy Solutions

- As before we assume that $(\mathbf{A_1}) - (\mathbf{A_2})$ are satisfied;
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Theorem (8) - (The Limit Profile of the Minimizers)

For any given sequence (ε_k) of positive numbers **converging to zero***, there exists a* **least energy solution** $\mathbf{u}_k = (u_{1k}, \ldots, u_{\ell k})$ *to the system $(\mathcal{S}^{\phi}_{\Omega,\varepsilon_k})$ with the following limit profile:*

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$$\lim_{k\to\infty} \|u_{ik} - w_i(\varepsilon_k^{-1}\,\cdot\,)\|_{\varepsilon_k} = 0 \quad \textit{for all} \;\; i = 1, \dots, \ell,$$

and $\lim\limits_{\varepsilon\to 0} c^{\ell}_{\varepsilon}(\Omega) = c^{\ell}_{\infty}.$

Moreover, $w_i \geq 0$ if $u_{ik} \geq 0$ for all $k \in \mathbb{N}$.

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(II) *If* **dim**$(\mathrm{Fix}(G)) > 0$, *then, for each* $i = 1, \dots, \ell$, *there exist a sequence* (ζ_{ik}) **in** $\Omega \cap \mathrm{Fix}(G)$ *and a* **least energy solution** v_i *to the problem* $(\mathcal{P}_i^{\phi_i})$ *such that, after passing to a subsequence,*

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$$\text{and} \quad \lim_{\varepsilon\to 0} c_\varepsilon^\ell(\Omega) = \sum_{i=1}^{\ell} c_i.$$

Moreover, $v_i \geq 0$ *if* $u_{ik} \geq 0$ *for all* $k \in \mathbb{N}$*.*

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- As before, $\widetilde{v}_1, \ldots, \widetilde{v}_m$ are also **positive** and since $(\mathcal{P}_i)$ has a **unique** positive radial solution v_i, any other positive solution is a translation of it;

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- As $\varepsilon_k \to 0$, we have that $\xi_{ik} \in B_1(0)$ for k large enough. From the corresponding statements for ζ_{ik} and $\widetilde{v}_i$ one derives the remaining conclusions involving the component functions;

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Continuation of the Proof of Theorem 4.

- As before, $\widetilde{v}_1, \dots, \widetilde{v}_m$ are also **positive** and since $(\mathcal{P}_i)$ has a **unique** positive radial solution v_i, any other positive solution is a translation of it;
- If $i \in \{m+1, \dots, \ell\}$, then $\widetilde{v}_i \in H^1(\mathbb{R}^N)^\phi$ with ϕ as in Example (i), i.e., $\widetilde{v}_i$ satisfies $\widetilde{v}_i(z_1, z_2, x) = \widetilde{v}_i(\mathrm{e}^{\mathrm{i}\vartheta} z_1, \mathrm{e}^{\mathrm{i}\vartheta} z_2, x)$ for all $\vartheta \in [0, 2\pi)$ and $\widetilde{v}_i(z_1, z_2, x) = -\widetilde{v}_i(z_2, z_1, x)$, for all $(z_1, z_2, x) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4}$;
- There exists $\vartheta_i \in \mathrm{Fix}(G)$ such that $v_i(y) := \widetilde{v}_i(y + \vartheta_i)$ satisfies (1.2);
- Let $\xi_{ik} := \zeta_{ik} + \varepsilon_k \vartheta_i$, so that $v_i(y) := \widetilde{v}_i(y + \vartheta_i)$ is radial if $i = 1, \dots, m$ and it satisfies (1.2) if $i = m+1, \dots, \ell$;
- As $\varepsilon_k \to 0$, we have that $\xi_{ik} \in B_1(0)$ for k large enough. From the corresponding statements for ζ_{ik} and $\widetilde{v}_i$ one derives the remaining conclusions involving the component functions;
- The inequality $\mathfrak{c}_m < \widehat{\mathfrak{c}}_m$ follows immediately from Proposition 6.

$\square$

Thank You For Your Attention!