Existence and nonexistence results for Lane-Emden type systems

Gabrielle Nornberg

Universidad de Chile

Seminário de EDPs da Unicamp - December 2, 2021

Joint works with Liliane Maia (UnB) and Filomena Pacella (La Sapienza)

- L. Maia, N., F. Pacella. *Existence, nonexistence and uniqueness for Lane-Emden type fully nonlinear systems*. arXiv:2107.05536.
- L. Maia, N., F. Pacella. *A dynamical system approach for Lane–Emden type problems*. Rio de Janeiro, Editora do IMPA, 33º CBM, 2021.
- L. Maia, N., F. Pacella, *A dynamical system approach to a class of radial weighted fully nonlinear equations*, CPDE, v.46, Issue 4, p. 573-610, 2021.

Outline

1 Motivation: scalar case

2 The dynamical system approach

3 Lane-Emden type system

The Lane-Emden equation

$$\begin{cases} \Delta u + |u|^{p-1}u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{LE}$$

where $p > 1$

Physics motivation

- Astrophysics: stability of stellar structures;
- Modeled as polytropic fluids from General Relativity, such that it holds a nonlinear relation between the mass density and the pressure.

The Lane-Emden equation: the radial case

$$\begin{cases} \Delta u + |u|^{p-1}u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{LE}$$

where $p > 1$

$$\operatorname{spect} D^2u(r) = \{u''(r), \tfrac{u'(r)}{r}, \ldots, \tfrac{u'(r)}{r}\}, \quad r = |x|,$$

$$u'' + \tfrac{N-1}{r}u' = -u^p, \quad u'(0) = 0, \quad u(0) = \gamma > 0 \tag{LE radial}$$

Hénon-Lane-Emden equation

$$\begin{cases} \Delta u + |x|^a |u|^{p-1} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{HLE}$$

where $p > 1$

Physics motivation

- Hénon (1973): Numerical analysis of the stability of spherically steady state stellar systems in the context of the so-called concentric shell model;
- General Relativity: back holes as Schwarzschild's stationary and spherically symmetric singular solution to Einstein's equations;
- Penrose (1965): black holes can form, at their heart they hide a singularity;
- Peebles (1972): distribution of stars near a massive collapsed object located at the cluster center;
- Nobel Prize 2019: black hole like objects perceived and photographed.

Hénon-Lane-Emden equation: the radial case

$$\begin{cases} \Delta u + |x|^a \, |u|^{p-1} u &= 0 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{HLE}$$

where $p > 1$, $a > -1$,

$$u'' + \frac{N-1}{r} u' = -r^a u^p, \quad u'(0) = 0, \quad u(0) = \gamma > 0 \qquad \text{(HLE radial)}$$

Classical change of variables

$r := s^\kappa$, $v(s) := \kappa^\alpha u(r)$, where $\kappa = \frac{2}{2+a}$, and so $\hat{N} = \frac{2(N+a)}{2+a}$.

$$u'' + \frac{\hat{N}-1}{r} u' = -u^p, \quad u'(0) = 0, \quad u(0) = \gamma > 0$$

The Pucci-Hénon-Lane-Emden equation

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) + |x|^a u^p = 0, \quad u > 0 \;\text{ in }\; \Omega, \tag{$P_\pm$}$$

where $a > -1$, $p > 1$, and for $0 < \lambda \leq \Lambda$,

$$\mathcal{M}^{+}_{\lambda,\Lambda}(X) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i \leq 0} e_i, \quad \mathcal{M}^{-}_{\lambda,\Lambda}(X) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i \leq 0} e_i.$$

Whenever Ω has a boundary, we set

$$u = 0 \text{ on } \partial\Omega, \quad \text{ or } \quad u = 0 \text{ on } \partial\Omega \setminus \{0\} \text{ if } u(0) = +\infty.$$

The radial setting

$$\mathcal{M}^{+}_{\lambda,\Lambda}(X) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i\leq 0} e_i, \quad \mathcal{M}^{-}_{\lambda,\Lambda}(X) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i\leq 0} e_i,$$

$$\operatorname{spect} D^2u(r) = \{u''(r), \tfrac{u'(r)}{r}, \ldots, \tfrac{u'(r)}{r}\},$$

$$u'' \;=\; M_+(-r^{-1}(N-1)\, m_+(u') - r^a u^p), \quad u>0; \qquad (P_+)$$

$$m_+(s) = \begin{cases} \lambda s & \text{if } s\leq 0 \\ \Lambda s & \text{if } s>0 \end{cases}, \quad M_+(s) = \begin{cases} s/\lambda & \text{if } s\leq 0 \\ s/\Lambda & \text{if } s>0; \end{cases}$$

The dimensional-like numbers as defined as:

$$\tilde{N}_+ = \frac{\lambda}{\Lambda}(N-1)+1, \qquad \tilde{N}_- = \frac{\Lambda}{\lambda}(N-1)+1.$$

Types of decays

Let u be a radial solution of $(P_\pm)$ for $\Omega = \mathbb{R}^N$. Set $r = |x|$ and $\alpha = \frac{2+a}{p-1}$. Then u is said to be:

(i) *fast decaying* if there exists $c > 0$ such that $\lim_{r\to\infty} r^{\tilde{N}-2}u(r) = c$, where $\tilde{N}$ is either $\tilde{N}_+$ if the operator is $\mathcal{M}^+$ or $\tilde{N}_-$ for $\mathcal{M}^-$;

(ii) *slow decaying* if there exists $c > 0$ such that $\lim_{r\to\infty} r^{\alpha}u(r) = c$;

(iii) *pseudo-slow decaying* if there exist constants $0 < c_1 < c_2$ such that

$$c_1 = \liminf_{r\to\infty} r^{\alpha}u(r) < \limsup_{r\to\infty} r^{\alpha}u(r) = c_2.$$

Theorems 1.1 and 1.2 in [Felmer-Quaas, AIHP 2003]

Assume $a = 0$, $\tilde{N}_+ > 2$, and $\lambda < \Lambda$. Then there exist critical exponents p_+^*, p_-^* satisfying the bounds

$$\max\left\{\frac{\tilde{N}_+}{\tilde{N}_+-2}, \frac{N+2}{N-2}\right\} < p_+^* < \frac{\tilde{N}_++2}{\tilde{N}_+-2} \quad \text{and} \quad \frac{\tilde{N}_-+2}{\tilde{N}_--2} < p_-^* < \frac{N+2}{N-2},$$

such that the following holds for $\Omega = \mathbb{R}^N$:

(i) if $p \in (1, p_\pm^*)$ there is no nontrivial radial solution of $(P_\pm)$;

(ii) if $p = p_\pm^*$ there exists a unique fast decaying radial solution to $(P_\pm)$;

(ii) If $p > p_\pm^*$ there exists a unique radial solution of $(P_\pm)$, which is either slow decaying or pseudo-slow decaying.

Theorems 1.1 and 1.2 in [Felmer-Quaas, AIHP 2003]

Assume $a = 0$, $\tilde{N}_+ > 2$, and $\lambda < \Lambda$. Then there exist critical exponents p_+^*, p_-^* satisfying the bounds

$$\max\left\{\frac{\tilde{N}_+}{\tilde{N}_+ - 2}, \frac{N+2}{N-2}\right\} < p_+^* < \frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2} \quad \text{and} \quad \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2} < p_-^* < \frac{N+2}{N-2},$$

such that the following holds for $\Omega = \mathbb{R}^N$:

(i) if $p \in (1, p_\pm^*)$ there is no nontrivial radial solution of $(P_\pm)$;

(ii) if $p = p_\pm^*$ there exists a unique fast decaying radial solution to $(P_\pm)$;

(ii) If $p > p_\pm^*$ there exists a unique radial solution of $(P_\pm)$, which is either slow decaying or pseudo-slow decaying.

The Hénon case

Attention: the classical change of variable does not work.

Outline

1 Motivation: scalar case

2 The dynamical system approach

3 Lane-Emden type system

A quadratic system

$$u'' \; = \; M_+(-r^{-1}(N-1)\, m_+(u') - r^a u^p), \quad u > 0; \qquad (P_+)$$

$$m_+(s) = \begin{cases} \lambda s & \text{if } s \le 0 \\ \Lambda s & \text{if } s > 0 \end{cases}, \quad M_+(s) = \begin{cases} s/\lambda & \text{if } s \le 0 \\ s/\Lambda & \text{if } s > 0; \end{cases}$$

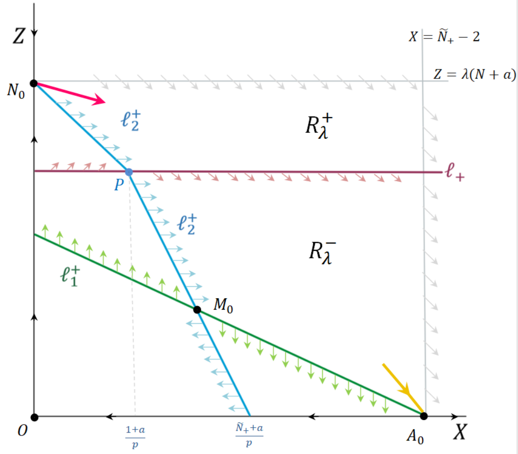
The new variables

$$X(t) = -\frac{ru'(r)}{u(r)}, \qquad Z(t) = -\frac{r^{1+a}\, u^p(r)}{u'(r)} \qquad \text{for } \; t = \ln(r),$$

$$\text{in } 1Q, \quad \begin{cases} \dot{X} &= X\,[X + 1 - M_+(\lambda(N-1) - Z)], \\ \dot{Z} &= Z\,[1 + a - pX + M_+(\lambda(N-1) - Z)]; \end{cases}$$

$$\text{in } 3Q, \quad \dot{X} = X\,[X - (\tilde{N}_- - 2) + Z/\Lambda], \;\; \dot{Z} = Z\,[\tilde{N}_- + a - pX - Z/\Lambda]$$

The flow in $1Q$ for $\mathcal{M}^+$: a priori bounds and blow-up



- Invariant axes, stationary points, the lines ℓ_+, ℓ_1^+ (fix), ℓ_2^+ (changing with p), A_0 is a saddle point, regular solutions from N_0, and $M_0 \in \operatorname{int} 1Q$ (case $p > p_+^{s,a}$).

The Laplacian operator: periodic orbits

$$p_\Delta^a = \frac{N+2+2a}{N-2}, \qquad \alpha = \frac{2+a}{p-1}.$$

Dulac's criterion [Bidaut-Verón, Giacomini 2010]

Let $\lambda = \Lambda = 1$, $p > 1$. There are no periodic orbits to $(DS)_+$ when $p \neq p_\Delta^a$.

Proof

Let $\varphi(X,Z) = X^\alpha Z^\beta$, where $\beta = \frac{3-p}{p-1}$ and $\alpha = \frac{2+a}{p-1}$. Then

$$\Phi := \partial_X(\varphi f) + \partial_Z(\varphi g) = \frac{\varphi(X,Z)}{p-1} \cdot [-p(N-2) + (N+2+2a)].$$

Indeed,

$$\int_{\partial D} \varphi \{f \,\mathrm{d}Z - g \,\mathrm{d}X\} = \int_D \Phi \,\mathrm{d}X\mathrm{d}Z.$$

Case $p < p_\Delta^a$

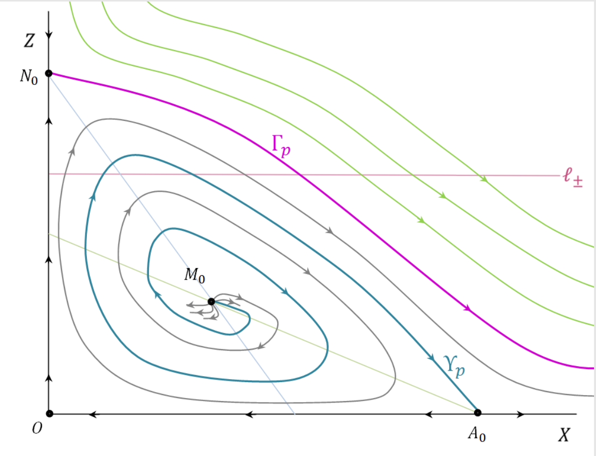


Figure: Case $p > \frac{N+a}{N-2}$, M_0 is a source, regular solutions in the ball.

Case $p > p^a_\Delta$

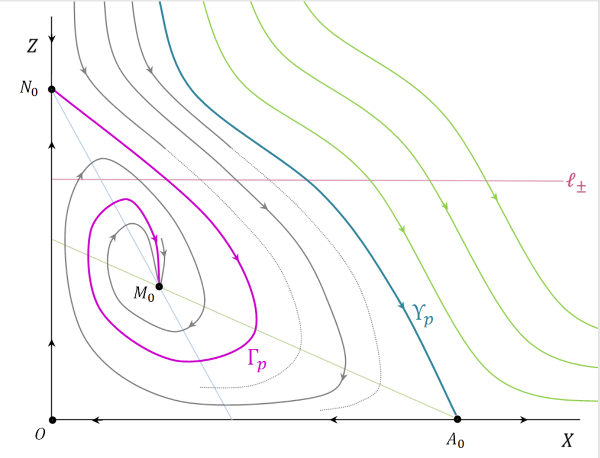


Figure: M_0 is a sink, regular solutions are slow decaying.

Case $p = p_\Delta^a$

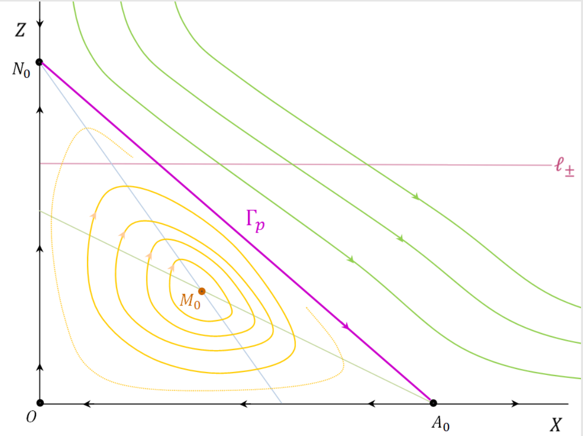


Figure: M_0 is a center, the regular solution is fast decaying.

Dulac range for $\mathcal{M}^{\pm}$

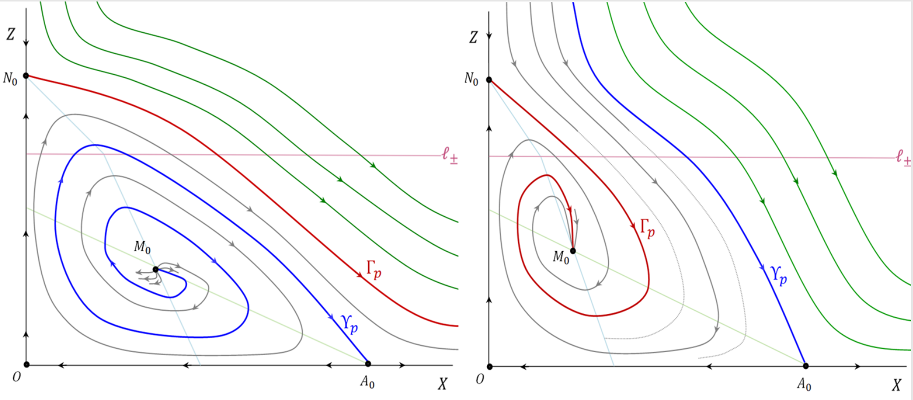


Figure: $p > p_{\pm}^{s,a}$, cases $p \in \mathcal{C}$ (LHS) and $p \in \mathcal{S}$ (RHS) without periodic orbits.

At the critical exponent for $\mathcal{M}^+$

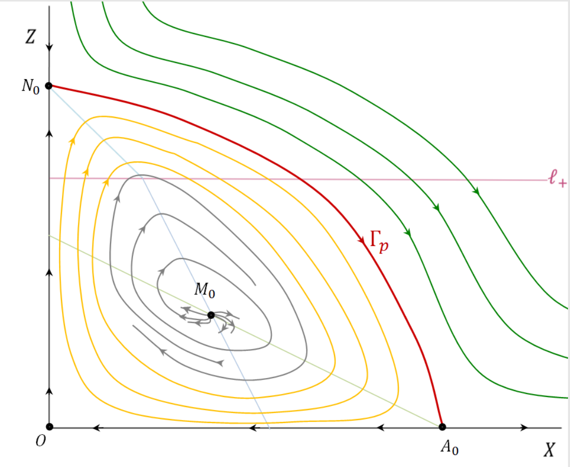
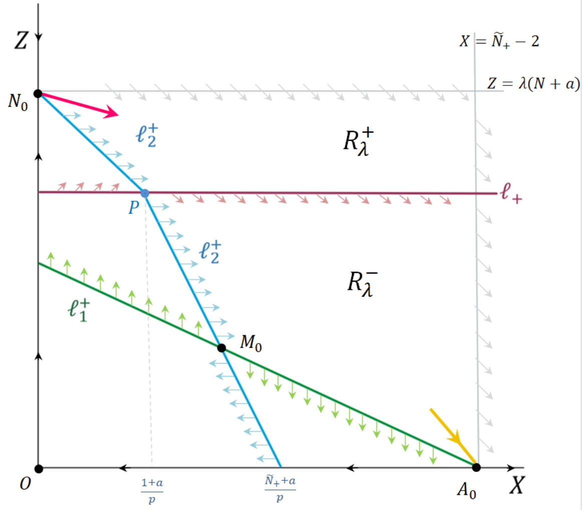


Figure: Case $p = p^*_{a+} \in \mathcal{F}$, M_0 is a source and $\Gamma_p = \Upsilon_p$.

Ordering of regular solutions



In the presence of a periodic orbit for $\mathcal{M}^+$

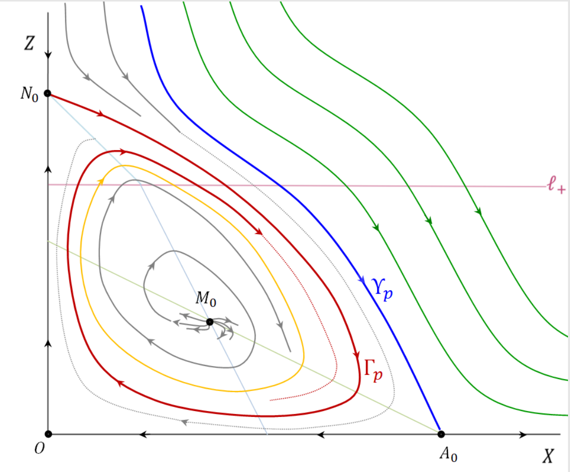


Figure: Case $p \in (p^*_{a+}, p^{p,a}_+)$, M_0 is a source.

In the presence of a periodic orbit for $\mathcal{M}^-$

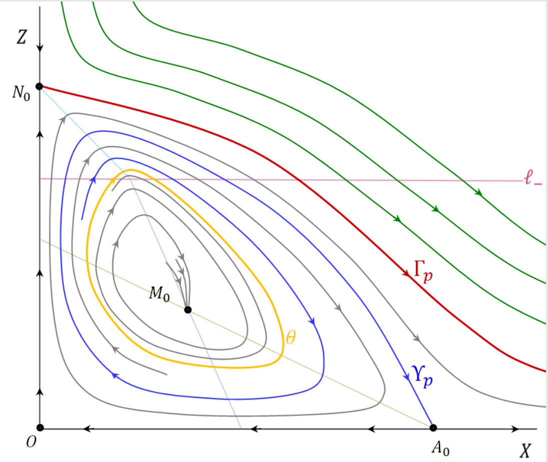


Figure: Case $p \in (p_-^{p,a}, p_{a-}^*)$, M_0 is a sink.

Some open problems...

- To find the precise range of periodic orbits for $\mathcal{M}^{\pm}$;
- To obtain an explicit "bubble" expression at $\mathcal{F}$;
- Are there nonradial positive solutions in $\mathbb{R}^N$ or $\mathbb{R}^N \setminus B_R$?

Outline

1 Motivation: scalar case

2 The dynamical system approach

3 Lane-Emden type system

The Lane-Emden system

$$\begin{cases} \Delta u + v^q = 0 & \text{in} \quad \mathbb{R}^N \\ \Delta v + u^p = 0 & \text{in} \quad \mathbb{R}^N \\ u,\ v > 0 & \text{in} \quad \mathbb{R}^N \end{cases}$$

where $p, q > 0$, $pq > 1$,

$$\frac{N}{p+1} + \frac{N}{q+1} = N - 2 \qquad \text{(H)}$$

Conjecture: Nonexistence in the subcritical and superlinear case

The Pucci-Lane-Emden system

$$\left\{ \begin{array}{rcll} \mathcal{M}^{\pm}(D^2u) + v^p & = & 0, & \text{in} \quad \Omega \\ \mathcal{M}^{\pm}(D^2v) + u^q & = & 0, & \text{in} \quad \Omega \\ u,\, v & > & 0 & \text{in} \quad \Omega \end{array} \right.$$

where $p, q > 0$, $pq > 1$.

- Quaas-Sirakov 2006; Armstrong-Sirakov 2009
- Symmetry in the ball: Moreira dos Santos, N. JDE 2020
- Eigenvalue problem at $pq = 1$: Moreira dos Santos, N., Schiera, Tavares 2020

The dynamical system

$$\text{for } \mathcal{M}^+ \text{ in } \mathcal{K}: \quad \begin{cases} u'' = M_+(-\lambda r^{-1}(N-1)\,u' - v^p), \\ v'' = M_+(-\lambda r^{-1}(N-1)\,v' - u^q), \quad u,\ v > 0; \end{cases}$$

The change of variables

$$X(t) = -\frac{ru'}{u},\ Y(t) = -\frac{rv'}{v},\ Z(t) = -\frac{rv^p}{u'},\ W(t) = -\frac{ru^q}{v'},\ t = \ln(r),$$

$$\mathcal{M}^+ \text{ in } \mathcal{K}: \quad \begin{cases} \dot{X} = X\,[\,X + 1 - M_+(\lambda(N-1) - Z)\,] \\ \dot{Y} = Y\,[\,Y + 1 - M_+(\lambda(N-1) - W)\,] \\ \dot{Z} = Z\,[1 - pY + M_+(\lambda(N-1) - Z)\,] \\ \dot{W} = W\,[1 - qX + M_+(\lambda(N-1) - W)\,]. \end{cases}$$

Existence x nonexistence for $\mathcal{M}^+$

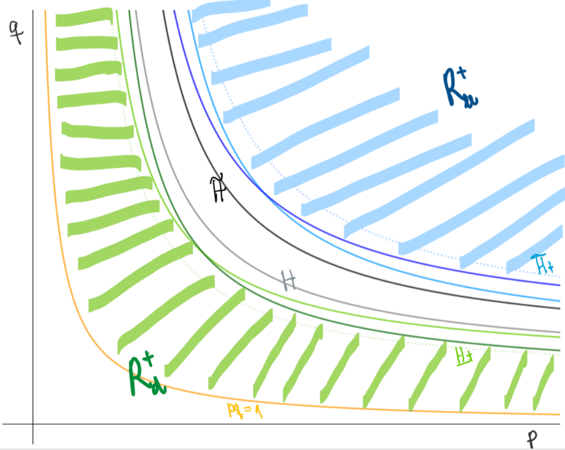


Figure: Our existence and nonexistence regions.

Liouville improvement

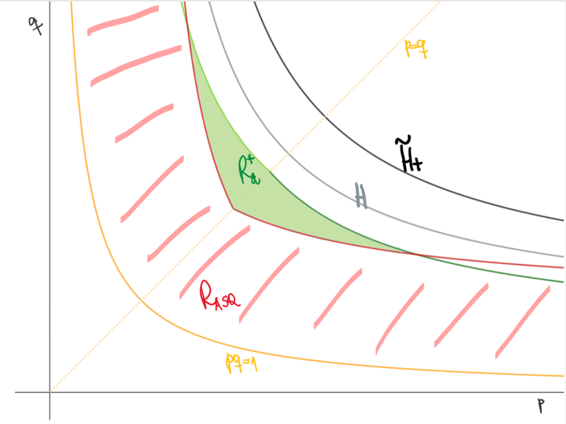


Figure: Our improved region. The asymptotic of the upper boundary of $\mathcal{R}_{\mathrm{ASQ}}$ is described by $\tilde{\mathcal{H}}_+$ when either $p \to +\infty$ or $q \to +\infty$.

The concavity of the solutions

Theorem

In $\mathcal{R}_d^{\pm}$ regular solutions of the Dirichlet problem (LE) in a ball change concavity only once.

- Nonexistence of oscillating solutions

The exclusion principle

We define

$$\mathcal{C} = \{(p,q) \in \mathbb{R}^2 :\ p,q > 0,\ pq > 1; \text{ there exists a solution } (u,v) \text{ of (LE) in } B_R,\ u = 0 \text{ on } \partial B_R\};$$

$$\mathcal{G} = \{(p,q) \in \mathbb{R}^2 :\ p,q > 0,\ pq > 1; \text{ there exists a radial solution } (u,v) \text{ of (LE) in } \mathbb{R}^N\}.$$

Theorem

For $\mathcal{C}$ and $\mathcal{G}$ as above concerning regular solutions, we have $\{(p,q) \in \mathbb{R}^2 :\ p,q > 0,\ pq > 1\} = \mathcal{C} \sqcup \mathcal{G}$, where $\sqcup$ is a disjoint union.

The critical in exterior domains

Exterior domain Neumann

Let $\lambda \leq \Lambda$ and $R > 0$. Regarding solutions of (LE) defined in the exterior domain $\mathbb{R}^N \setminus B_R$, with $u, v > 0$ on ∂B_R, and $\partial_\nu u, \partial_\nu v = 0$ on ∂B_R, it holds:

(i) if $(p, q) \in \overline{\mathcal{R}_D^{\pm}}$ then there is no radial positive exterior domain solution of (LE);

(ii) if $(p, q) \in \mathcal{R}_u^{\pm}$ there exist exterior domain fast decaying solutions of (LE).

Moreover, if $\lambda = \Lambda$ then the hyperbola (H) in gives us the threshold for existence and nonexistence of exterior domain solutions with Neumann boundary condition.

The critical in exterior domains

Liouville for exterior domains Dirichlet

Let $\lambda = \Lambda$. If $\alpha + \beta \geq N - 2$ then there is no radial positive exterior domain solution of (LE) for any $R > 0$ with Dirichlet boundary condition $u = 0$ on ∂B_R.

Energy methods

Consider $\lambda = \Lambda$.

$$E(r) = r^N \left(u'v' + \frac{1}{\lambda}\frac{v^{p+1}}{p+1} + \frac{1}{\lambda}\frac{u^{q+1}}{q+1} + \frac{N}{p+1}\frac{vu'}{r} + \frac{N}{q+1}\frac{uv'}{r} \right)$$

Equivalently, in terms of the variables X, Y, Z, W,

$$E(t) = e^{t(N-2-\alpha-\beta)}(XZ)^{\frac{\beta}{2}}(YW)^{\frac{\alpha}{2}}$$
$$\left(XY + \frac{XZ}{\lambda(p+1)} + \frac{YW}{\lambda(q+1)} - \frac{NX}{p+1} - \frac{NY}{q+1} \right)$$

[B-V,G] Bidaut-Veron, Giacomini approach ADE 2010.

Energy methods

If $\lambda = \Lambda$,

$$E'(r) = r^{N-1}\, u' v' \, \left\{ \frac{N}{p+1} + \frac{N}{q+1} - (N-2) \right\}.$$

Now, since regular positive solutions satisfy $u', v' < 0$, it follows that

$$\begin{cases} \dot{E} < 0 & \text{above (H)} \\ \dot{E} = 0 & \text{on (H)} \\ \dot{E} > 0 & \text{below (H)} \end{cases}$$

Energy por the Pucci system

$$E(r) = \begin{cases} r^N \left(u'v' + \frac{1}{\lambda}\frac{v^{p+1}}{p+1} + \frac{1}{\lambda}\frac{u^{q+1}}{q+1} + \frac{N}{p+1}\frac{vu'}{r} + \frac{N}{q+1}\frac{uv'}{r} \right) & \text{in } \{u''<0\}\cap\{v''<0\}, \\ r^\sigma \left(u'v' + \frac{1}{\lambda}\frac{v^{p+1}}{p+1} + \frac{1}{\Lambda}\frac{u^{q+1}}{q+1} + \frac{N}{p+1}\frac{vu'}{r} + \frac{\tilde{N}_+}{q+1}\frac{uv'}{r} \right) & \text{in } \{u''<0\}\cap\{v''>0\}, \\ r^\sigma \left(u'v' + \frac{1}{\Lambda}\frac{v^{p+1}}{p+1} + \frac{1}{\lambda}\frac{u^{q+1}}{q+1} + \frac{\tilde{N}_+}{p+1}\frac{vu'}{r} + \frac{N}{q+1}\frac{uv'}{r} \right) & \text{in } \{u''>0\}\cap\{v''<0\}, \\ r^{\tilde{N}_+} \left(u'v' + \frac{1}{\Lambda}\frac{v^{p+1}}{p+1} + \frac{1}{\Lambda}\frac{u^{q+1}}{q+1} + \frac{\tilde{N}_+}{p+1}\frac{vu'}{r} + \frac{\tilde{N}_+}{q+1}\frac{uv'}{r} \right) & \text{in } \{u''>0\}\cap\{v''>0\}. \end{cases}$$

Energy for the Pucci system

$$E'(r) = \begin{cases} r^{N-1}\, u' v' \left\{ \frac{N}{p+1} + \frac{N}{q+1} - (N-2) \right\} & \text{in } R^+_{\lambda,Z} \cap R^+_{\lambda,W}, \\ r^{\tilde{N}-1}\, u' v' \left\{ \frac{\tilde{N}_\pm}{p+1} + \frac{\tilde{N}_\pm}{q+1} - (\tilde{N}_\pm - 2) \right\} & \text{in } R^-_{\lambda,Z} \cap R^-_{\lambda,W}. \end{cases}$$

$$E'(r) = \begin{cases} r^{\sigma-1}\, u' v' \,(2 - N - \tilde{N}_+ + \sigma + \frac{N}{p+1} + \frac{\tilde{N}_+}{q+1}) + r^{\sigma-1}\, \mathcal{E}_{\sigma,1}(r) & \text{in } R^+_{\lambda,Z} \cap R^-_{\lambda,W}, \\ r^{\sigma-1}\, u' v' \,(2 - N - \tilde{N}_+ + \sigma + \frac{\tilde{N}_+}{p+1} + \frac{N}{q+1}) + r^{\sigma-1}\, \mathcal{E}_{\sigma,2}(r) & \text{in } R^-_{\lambda,Z} \cap R^+_{\lambda,W}, \end{cases}$$

where

$$\mathcal{E}_{\sigma,1}(r) = \left\{ \tfrac{v^{p+1}}{\lambda(p+1)} + \tfrac{N v u'}{r(p+1)} \right\} (\sigma - N) + \left\{ \tfrac{u^{q+1}}{\Lambda(q+1)} + \tfrac{\tilde{N}_+ v' u}{r(q+1)} \right\} (\sigma - \tilde{N}_+),$$

$$\mathcal{E}_{\sigma,2}(r) = \left\{ \tfrac{v^{p+1}}{\Lambda(p+1)} + \tfrac{\tilde{N}_+ v u'}{r(p+1)} \right\} (\sigma - \tilde{N}_+) + \left\{ \tfrac{u^{q+1}}{\lambda(q+1)} + \tfrac{N v' u}{r(q+1)} \right\} (\sigma - N),$$

Regular solutions for the system involving the Laplacian operator [B-V, G]

Step 1) Existence of ground state solutions in $\overline{\mathcal{R}}_u^+$

- E is nonincreasing for all r in $\mathcal{R}_u^+$.
- If there not existed any regular solution in $\mathbb{R}^N$, then there would existed a regular solution (u, v) in the ball B_R, with trajectory $\Gamma(t) = (X(t), Y(t), Z(t), W(t))$ s.t. $X(T) = Y(T) = +\infty$ at $T = \ln(R)$.
- (u, v) starts at $r = 0$ with zero energy. Then the energy of (u, v) remains nonpositive whenever it is defined, in particular at the limit point.
- At $r = R$ we have $E(R) = r^{\tilde{N}_+} u'(R)v'(R) > 0$, contradiction with $E(R) \leq 0$. Then there exists a solution of (LE) in $\mathbb{R}^N$.

Regular solutions for the system involving the Laplacian operator [B-V, G]

Step 2) Nonexistence of ground states in $\mathcal{R}_d^+$

- E is a nondecreasing when p, q lies in the region $\mathcal{R}_d^+$.
- If there existed a nontrivial positive regular solution in $\mathbb{R}^N$, the correspondent regular trajectory Γ would start at $-\infty$ with zero energy from N_0.
- $E(r) \leq Cr^{N-2-\alpha-\beta}$ for large r. Below (H) one has $\lim_{r\to\infty} E(r) = 0$. But this contradicts the monotonicity of E in the case of a nontrivial pair solution u, v in $\mathbb{R}^N$.
- Moreover, there exists a solution of the Dirichlet problem in a ball.

Liouville for exterior domain solutions of the system with the Laplacian

$$\mathbb{E}(r) = r^{\tilde{N}_+}\left(u'v' + \frac{1}{\Lambda}\frac{v^{p+1}}{p+1} + \frac{1}{\Lambda}\frac{u^{q+1}}{q+1} + \mathfrak{A}\frac{vu'}{r} + \mathfrak{B}\frac{uv'}{r}\right)$$

- Assume by contradiction there exists a nontrivial exterior domain solution of (LE) in $\mathbb{R}^N \setminus B_R$, with $u, v = 0$ on ∂B_R.
- Since $N = \tilde{N}_+$, by taking $\delta \geq 0$ so that $\frac{N}{p+1} + \frac{N}{q+1} = N - 2 + \delta$, with $\mathfrak{A} = \frac{N}{p+1} - \frac{\delta}{2}$ and $\mathfrak{B} = \frac{N}{q+1} - \frac{\delta}{2}$, then we get

$$\mathbb{E}'(r) = \frac{\delta r^{N-1}}{2\Lambda}(v^{p+1} + u^{q+1}) \geq 0 \text{ for all } r \geq R.$$

- Since $\mathbb{E}$ is increasing and $\mathbb{E}(\Upsilon, R) > 0$, we have $\mathbb{E} > 0$ for all $r \geq R$. As in the Neumann case, Υ cannot approach neither A_0, P_0, Q_0 nor M_0 at infinity because they have zero energy; while Υ is not oscillating at infinity. Hence, the existence of Υ is denied.

Open problems

We recall

$$\mathcal{C} = \{(p,q) \in \mathbb{R}^2 : \ p,q > 0, \ pq > 1; \ \exists \text{ a solution } (u,v) \text{ of (LE) in } B_R, \ u = 0 \text{ on } \partial B_R\}$$

$$\mathcal{G} = \{(p,q) \in \mathbb{R}^2 : \ p,q > 0, \ pq > 1; \ \exists \text{ a radial solution } (u,v) \text{ of (LE) in } \mathbb{R}^N\}$$

Remark: $\partial\mathcal{C} \subset \mathcal{G}$ since $\mathcal{C}$ is open

Conjectures in the fully nonlinear radial case:

(1) $\partial\mathcal{C}$ turns out to be a critical curve on the (p,q) plane, being the threshold between existence and nonexistence of regular solutions in $\mathbb{R}^N$;

(2) It is also a critical curve for existence and nonexistence of fast decaying exterior domain solutions;

(3) The behavior of the solutions at infinity is subject to a fast decaying profile.

Thank you for your attention!

gnornberg@dim.uchile.cl

sites.google.com/view/gabrielle-nornberg