Regularity estimates for the parabolic normalized p-Laplace equation

Jointly w/ M. Santos

Pêdra Andrade

Seminário de EDPs - IMECC

ICMC - USP
OUT 28 2021

Outline

1. Motivation;
2. Previous developments;
3. Approximation methods;
4. Improved regularity in Hölder spaces;
5. Regularity estimates in Sobolev spaces;
6. Sketch of the proof.

Motivation

Motivation

For $1 < p < \infty$, we consider the *p-Laplace equation*

$$\Delta_p u := \operatorname{div}(|Du|^{p-2} Du) = 0$$

Motivation

For $1 < p < \infty$, we consider the p-Laplace equation

$$\Delta_p u := \operatorname{div}(|Du|^{p-2}Du) = 0,$$

is the Euler-Lagrange equation associated to the functional

$$\frac{1}{p}\int |Du|^p$$

Motivation

For $1 < p < \infty$, we consider the *p*-Laplace equation

$$\Delta_p u := \mathrm{div}(|Du|^{p-2} Du) = 0,$$

is the Euler-Lagrange equation associated to the functional

$$\frac{1}{p} \int |Du|^p.$$

In this case, we pointed out well-known properties:

1. The equation is in divergence form

Motivation

For $1 < p < \infty$, we consider the *p*-Laplace equation

$$\Delta_p u := \operatorname{div}(|Du|^{p-2} Du) = 0,$$

is the Euler-Lagrange equation associated to the functional

$$\frac{1}{p} \int |Du|^p.$$

In this case, we pointed out well-known properties:

1. The equation is in divergence form;
2. The variational structure of the equation

Motivation

For $1 < p < \infty$, we consider the *p*-Laplace equation

$$\Delta_p u := \mathrm{div}(|Du|^{p-2} Du) = 0,$$

is the Euler-Lagrange equation associated to the functional

$$\frac{1}{p} \int |Du|^p.$$

In this case, we pointed out well-known properties:

1. The equation is in divergence form;
2. The variational structure of the equation;
3. Every weak solution of p-Laplace equation in the distribution sense is $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$.

Motivation

We study regularity theory for the viscosity solutions to

$$u_t(x,t) - \Delta_p^N u(x,t) = f(x,t) \quad \text{in} \quad Q_1,$$

Motivation

We study regularity theory for the viscosity solutions to

$$u_t(x,t) - \Delta_p^N u(x,t) = f(x,t) \quad \text{in} \quad Q_1,$$

where

1. The normalized p-Laplace operator is defined by

$$\begin{aligned}
\Delta_p^N u &:= |Du|^{2-p} \Delta_p u \\
&= \Delta u + (p-2)\Delta_\infty u \\
&= \Delta u + (p-2)\left\langle D^2 u \tfrac{Du}{|Du|}, \tfrac{Du}{|Du|} \right\rangle
\end{aligned}$$

Motivation

We study regularity theory for the viscosity solutions to

$$u_t(x,t) - \Delta_p^N u(x,t) = f(x,t) \quad \text{in} \quad Q_1,$$

where

1. The normalized p-Laplace operator is defined by

$$\begin{aligned} \Delta_p^N u &:= |Du|^{2-p}\Delta_p u \\ &= \Delta u + (p-2)\Delta_\infty u \\ &= \Delta u + (p-2)\left\langle D^2 u \tfrac{Du}{|Du|}, \tfrac{Du}{|Du|} \right\rangle; \end{aligned}$$

2. The right-hand side f is a continuous and bounded function in Q_1 and $1 < p < \infty$.

Motivation

Some remarks:

1. The equation above can be seen as a uniformly parabolic model in nondivergence form, with constants $\min\{(p-1,1)\}$ and $\max\{(p-1),1\}$

Motivation

Some remarks:

1. The equation above can be seen as a uniformly parabolic model in nondivergence form, with constants $\min\{(p-1,1)\}$ and $\max\{(p-1),1\}$;
2. This model has a singularity on the set $\{Du=0\}$, which implies that classical $\mathcal{C}^{1+\alpha}$-regularity can not be applied directly.

Motivation

The normalized p-Laplace equation appears in many branches of mathematics.

- For $p \to 1$,

$$u_t(x,t) - \Delta_1^N u(x,t) = 0$$

is the level set formulation of the mean curvature flow.

Motivation

The normalized p-Laplace equation appears in many branches of mathematics.

- For $p \to 1$,

$$u_t(x,t) - \Delta_1^N u(x,t) = 0$$

is the level set formulation of the mean curvature flow.

- For $1 < p < \infty$,

$$u_t(x,t) - \Delta_p^N u(x,t) = f(x,t)$$

This equation arises from tug-of-war stochastic games with noise.

Motivation

The normalized p-Laplace equation appears in many branches of mathematics.

- For $p \to 1$,

$$u_t(x,t) - \Delta_1^N u(x,t) = 0$$

is the level set formulation of the mean curvature flow.

- For $1 < p < \infty$,

$$u_t(x,t) - \Delta_p^N u(x,t) = f(x,t)$$

This equation arises from tug-of-war stochastic games with noise.

- For $p \to \infty$,

$$u_t(x,t) - \Delta_\infty^N u(x,t) = 0$$

This equation finds an application in image processing.

Viscosity solutions

Let $1 < p < \infty$ and f is a continuous function in Q_1. We say that $u \in \mathcal{C}(Q_1)$ is a viscosity subsolution to

$$u_t(x,t) - \Delta_p^N u(x,t) = f(x,t)$$

if, for every $(x_0, t_0) \in Q_1$ and $\varphi \in \mathcal{C}^2(Q_1)$ such that $u - \varphi$ has a local maximum at (x_0, t_0), we have

$$\begin{array}{ll}
\varphi_t(x_0,t_0) - \Delta_p^N \varphi(x_0,t_0) \le f(x_0,t_0), & \text{if } D\varphi(x_0,t_0) \neq 0 \\
\varphi_t(x_0,t_0) - \Delta\varphi(x_0,t_0) - (p-2)\lambda_{\max}(D^2\varphi(x_0,t_0)) & \\
\le f(x_0,t_0), & \text{if } D\varphi(x_0,t_0) = 0 \text{ and } p \ge 2. \\
\varphi_t(x_0,t_0) - \Delta\varphi(x_0,t_0) - (p-2)\lambda_{\min}(D^2\varphi(x_0,t_0)) & \\
\le f(x_0,t_0), & \text{if } D\varphi(x_0,t_0) = 0 \text{ and } 1 < p < 2.
\end{array}$$

Viscosity solution

Conversely, we say that $u \in \mathcal{C}(Q_1)$ is a viscosity supersolution to

$$u_t(x,t) - \Delta_p^N u(x,t) = f(x,t)$$

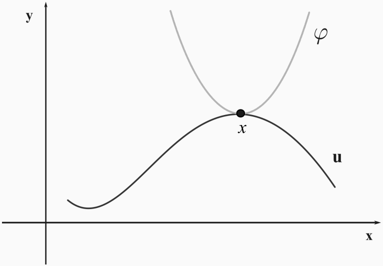
if, $(x_0,t_0) \in Q_1$ and $\varphi \in \mathcal{C}^2(Q_1)$ such that $u - \varphi$ has a local minimum at (x_0,t_0), we have

$$\begin{aligned}
&\varphi_t(x_0,t_0) - \Delta_p^N \varphi(x_0,t_0) \geq f(x_0,t_0), && \text{if } D\varphi(x_0,t_0) \neq 0\\
&\varphi_t(x_0,t_0) - \Delta\varphi(x_0,t_0) - (p-2)\lambda_{\min}(D^2\varphi(x_0,t_0)) && \\
&\geq f(x_0,t_0), && \text{if } D\varphi(x_0,t_0) = 0 \text{ and } p \geq 2.\\
&\varphi_t(x_0,t_0) - \Delta\varphi(x_0,t_0) - (p-2)\lambda_{\max}(D^2\varphi(x_0,t_0)) && \\
&\geq f(x_0,t_0), && \text{if } D\varphi(x_0,t_0) = 0 \text{ and } 1 < p < 2.
\end{aligned}$$

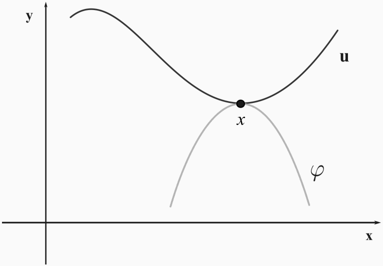
We call u is a viscosity solution, when it is subsolution and supersolution.

Viscosity Solutions

Viscosity Subsolution



Viscosity Supersolution



Previous developments

Manfredi, Parviainen and Rossi (2010): The authors characterized solutions to

$$u_t(x,t) - \Delta_p^N u(x,t) = 0,$$

in terms of an asymptotic mean value property. They also established the existence of solutions to using game-theoretic arguments.

Previous developments

Manfredi, Parviainen and Rossi (2010): The authors characterized solutions to

$$u_t(x,t) - \Delta_p^N u(x,t) = 0,$$

in terms of an asymptotic mean value property. They also established the existence of solutions to using game-theoretic arguments.

Banerjee and Garofalo (2013): They proved existence of the solutions to

$$u_t(x,t) - \Delta_p^N u(x,t) = 0,$$

and uniqueness by using comparison principles.

Previous developments

Parviainen and Ruosteenoja (2016): The authors obtained local Hölder and Lipschitz estimates to

$$u_t(x,t) - \Delta u - (p(x,t) - 2)\Delta_\infty^N u(x,t) = 0,$$

By using a game theoretic method for the case $p = p(x,t) > 2$.

Previous developments

Parviainen and Ruosteenoja (2016): The authors obtained local Hölder and Lipschitz estimates to

$$u_t(x,t) - \Delta u - (p(x,t) - 2)\Delta_\infty^N u(x,t) = 0,$$

By using a game theoretic method for the case $p = p(x,t) > 2$.

Attouchi and Parviainen (2018): The authors showed that viscosity solutions of

$$u_t - \Delta u - (p-2)\left\langle D^2 u \frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|}\right\rangle = f,$$

1. The solutions are of class $\mathcal{C}^{\beta,\beta/2}$ with $0 < \beta < 1$ and $\xi \in \mathbb{R}^d$

Previous developments

Parviainen and Ruosteenoja (2016): The authors obtained local Hölder and Lipschitz estimates to

$$u_t(x,t) - \Delta u - (p(x,t)-2)\Delta_\infty^N u(x,t) = 0,$$

By using a game theoretic method for the case $p = p(x,t) > 2$.

Attouchi and Parviainen (2018): The authors showed that viscosity solutions of

$$u_t - \Delta u - (p-2)\left\langle D^2 u \frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|}\right\rangle = f,$$

1. The solutions are of class $\mathcal{C}^{\beta,\beta/2}$ with $0 < \beta < 1$ and $\xi \in \mathbb{R}^d$;
2. For all $t \in [-r^2, 0]$, if $\operatorname{osc}_{B_r} u(\cdot,t) \leq A$, we obtain that the oscillation of u in Q_r is bounded from above by $CA + 4r^2\|f\|_{L^\infty(Q_1)}$.

Høeg and Lindqvist (2020): The authors proved that viscosity solutions of

$$u_t(x,t) - \Delta_p^N u(x,t) = 0,$$

are locally of class $W^{2,1;2}$ when $p \in (\frac{6}{5}, \frac{14}{5})$

Høeg and Lindqvist (2020): The authors proved that viscosity solutions of

$$u_t(x,t) - \Delta_p^N u(x,t) = 0,$$

are locally of class $W^{2,1;2}$ when $p \in (\frac{6}{5}, \frac{14}{5})$.

Dong, Peng, Zhang and Zhou (2020): The authors showed that viscosity solutions of

$$u_t(x,t) - \Delta_p^N u(x,t) = 0,$$

are of class $W^{2,1;q}$ for $q < 2 + \delta_{n,p}$ with $\delta_{n,p} \in (0,1)$, when $p \in (1,2) \cup \left(2, 3 + \frac{2}{d-2}\right)$.

Approximation methods

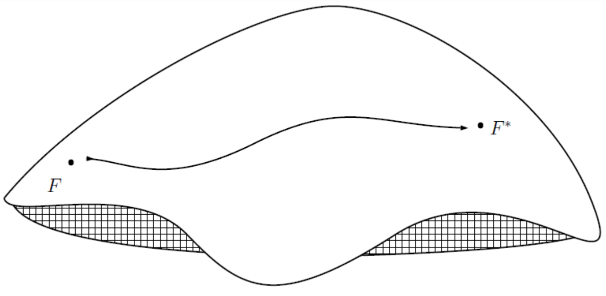
Approximation methods

General idea of the method

The strategy is to create a path connecting the studied equation for a nice limiting equation. Thus, we transport these great properties to the studied operator through this path

General idea of the method

The strategy is to create a path connecting the studied equation for a nice limiting equation. Thus, we transport these great properties to the studied operator through this path.



Improved regularity in Hölder spaces

Approximation methods

Approximation Lemma Let $u \in \mathcal{C}(Q_1)$ be a normalized viscosity solution to

$$u_t - \Delta u - (p-2)\left\langle D^2 u \frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|} \right\rangle = f \ \text{ in } \ Q_1$$

Approximation methods

Approximation Lemma Let $u \in \mathcal{C}(Q_1)$ be a normalized viscosity solution to

$$u_t - \Delta u - (p-2)\left\langle D^2 u \frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|} \right\rangle = f \text{ in } Q_1,$$

where $\xi \in \mathbb{R}^d$ and $f \in L^\infty(Q_1) \cap \mathcal{C}(Q_1)$. Given $\delta > 0$, there exists $\varepsilon > 0$ such that, if

$$\|f\|_{L^\infty(Q_1)} + |p-2| < \varepsilon$$

Approximation Lemma Let $u \in \mathcal{C}(Q_1)$ be a normalized viscosity solution to

$$u_t - \Delta u - (p-2)\left\langle D^2 u \frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|} \right\rangle = f \text{ in } Q_1,$$

where $\xi \in \mathbb{R}^d$ and $f \in L^\infty(Q_1) \cap \mathcal{C}(Q_1)$. Given $\delta > 0$, there exists $\varepsilon > 0$ such that, if

$$\|f\|_{L^\infty(Q_1)} + |p-2| < \varepsilon,$$

then we can find $h \in \mathcal{C}^{2,1}(Q_{7/9})$ such that

$$\sup_{Q_{7/9}} |u(x,t) - h(x,t)| < \delta.$$

Approximation methods

The result follows from two main ingredients:

1. Contradiction argument

Approximation methods

The result follows from two main ingredients:

1. Contradiction argument;
2. Stability result

Approximation methods

The result follows from two main ingredients:

1. Contradiction argument;
2. Stability result.

Consequences:

1. The solutions of the normalized p-Laplace are close to the solutions of the Heat equation.

Regularity in Hölder Spaces with respect to the spatial variable

Theorem (A. -Santos) Let $u \in \mathcal{C}(Q_1)$ be a normalized viscosity solution of

$$u_t - \Delta u - (p-2)\left\langle D^2 u \frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|} \right\rangle = f \ \text{ in } \ Q_1$$

Regularity in Hölder Spaces with respect to the spatial variable

Theorem (A. -Santos) Let $u \in \mathcal{C}(Q_1)$ be a normalized viscosity solution of

$$u_t - \Delta u - (p-2)\left\langle D^2 u \frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|} \right\rangle = f \text{ in } Q_1,$$

where $\xi \in \mathbb{R}^d$ and $f \in L^\infty(Q_1) \cap \mathcal{C}(Q_1)$. Then, given $\alpha \in (0,1)$, there exists $\varepsilon > 0$ such that

$$\|f\|_{L^\infty(Q_1)} + |p-2| \leq \varepsilon$$

Regularity in Hölder Spaces with respect to the spatial variable

Theorem (A. -Santos) Let $u \in \mathcal{C}(Q_1)$ be a normalized viscosity solution of

$$u_t - \Delta u - (p-2)\left\langle D^2 u \frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|}\right\rangle = f \;\text{ in }\; Q_1,$$

where $\xi \in \mathbb{R}^d$ and $f \in L^\infty(Q_1) \cap \mathcal{C}(Q_1)$. Then, given $\alpha \in (0,1)$, there exists $\varepsilon > 0$ such that

$$\|f\|_{L^\infty(Q_1)} + |p-2| \leq \varepsilon,$$

we can find a constant $0 < \rho < 1$ and a sequence of affine functions $(\ell_n)_{n \in N}$ of the form $\ell_n(x,t) := a_n + b_n \cdot x$ satisfying

$$\sup_{Q_{\rho^n}} |u(x,t) - \ell_n(x,t)| \leq \rho^{n(1+\alpha)}$$

Regularity in Hölder Spaces with respect to the spatial variable

Theorem (A. -Santos) Let $u \in \mathcal{C}(Q_1)$ be a normalized viscosity solution of

$$u_t - \Delta u - (p-2)\left\langle D^2 u \frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|} \right\rangle = f \ \text{ in } \ Q_1,$$

where $\xi \in \mathbb{R}^d$ and $f \in L^\infty(Q_1) \cap \mathcal{C}(Q_1)$. Then, given $\alpha \in (0,1)$, there exists $\varepsilon > 0$ such that

$$\|f\|_{L^\infty(Q_1)} + |p-2| \leq \varepsilon,$$

we can find a constant $0 < \rho < 1$ and a sequence of affine functions $(\ell_n)_{n \in N}$ of the form $\ell_n(x,t) := a_n + b_n \cdot x$ satisfying

$$\sup_{Q_{\rho^n}} |u(x,t) - \ell_n(x,t)| \leq \rho^{n(1+\alpha)},$$

$$|a_{n+1} - a_n| \leq C\rho^{n(1+\alpha)} \quad \text{and} \quad |b_{n+1} - b_n| \leq C\rho^{n\alpha}$$

for a constant $C > 0$ and for every $n \in \mathbb{N}$.

Ideas behind the proof and some consequences

The result follows from three main ingredients:

1. Induction argument

Ideas behind the proof and some consequences

The result follows from three main ingredients:

1. Induction argument;
2. Approximation Lemma

Ideas behind the proof and some consequences

The result follows from three main ingredients:

1. Induction argument;
2. Approximation Lemma;
3. Iteration process

Ideas behind the proof and some consequences

The result follows from three main ingredients:

1. Induction argument;
2. Approximation Lemma;
3. Iteration process.

Consequences to regularity theory:

1. The gradient is almost Lipschitz continuous by geometric methods

Ideas behind the proof and some consequences

The result follows from three main ingredients:

1. Induction argument;
2. Approximation Lemma;
3. Iteration process.

Consequences to regularity theory:

1. The gradient is almost Lipschitz continuous by geometric methods;

Sketch of the proof

Take $\delta > 0$, it follows from the Approximation Lemma that there exists function $h \in \mathcal{C}^{2,1}(Q_{8/9})$ such that

$$\sup_{Q_\rho} |u(x,t) - h(x,t)| \leq \delta$$

Sketch of the proof

Take $\delta > 0$, it follows from the Approximation Lemma that there exists function $h \in \mathcal{C}^{2,1}(Q_{8/9})$ such that

$$\sup_{Q_\rho} |u(x,t) - h(x,t)| \leq \delta.$$

Set

$$\ell(x,t) := h(0,0) + Dh(0,0) \cdot x$$

Sketch of the proof

Take $\delta > 0$, it follows from the Approximation Lemma that there exists function $h \in \mathcal{C}^{2,1}(Q_{8/9})$ such that

$$\sup_{Q_\rho} |u(x,t) - h(x,t)| \leq \delta.$$

Set

$$\ell(x,t) := h(0,0) + Dh(0,0) \cdot x.$$

By using regularity results for h, we have that

$$\sup_{Q_\rho} |h(x,t) - \ell(x,t)| \leq C\rho^2$$

Sketch of the proof

Take $\delta > 0$, it follows from the Approximation Lemma that there exists function $h \in \mathcal{C}^{2,1}(Q_{8/9})$ such that

$$\sup_{Q_\rho} |u(x,t) - h(x,t)| \leq \delta.$$

Set

$$\ell(x,t) := h(0,0) + Dh(0,0) \cdot x.$$

By using regularity results for h, we have that

$$\sup_{Q_\rho} |h(x,t) - \ell(x,t)| \leq C\rho^2,$$

From the triangular inequality, we obtain

$$\sup_{Q_\rho} |u(x,t) - \ell(x,t)| \leq \delta + C\rho^2$$

Sketch of the proof

Take $\delta > 0$, it follows from the Approximation Lemma that there exists function $h \in \mathcal{C}^{2,1}(Q_{8/9})$ such that

$$\sup_{Q_\rho} |u(x,t) - h(x,t)| \le \delta.$$

Set

$$\ell(x,t) := h(0,0) + Dh(0,0) \cdot x.$$

By using regularity results for h, we have that

$$\sup_{Q_\rho} |h(x,t) - \ell(x,t)| \le C\rho^2,$$

From the triangular inequality, we obtain

$$\sup_{Q_\rho} |u(x,t) - \ell(x,t)| \le \delta + C\rho^2.$$

Making universal choices, we define

$$\rho := (1/2C)^{\frac{1}{1-\alpha}} \quad \text{and} \quad \delta := \rho^{1+\alpha}/2.$$

Sketch of the proof

Assume that the case $n = k$ has been verified

Sketch of the proof

Assume that the case $n = k$ has been verified. We prove the case $n = k + 1$

Sketch of the proof

Assume that the case $n = k$ has been verified. We prove the case $n = k + 1$. Define $v_k : Q_1 \to \mathbb{R}^d$ such that

$$v_k(x,t) := \frac{u(\rho^k x, \rho^{2k} t) - \ell_k(\rho^k x, \rho^{2k} t)}{\rho^{k(1+\alpha)}}$$

Sketch of the proof

Assume that the case $n = k$ has been verified. We prove the case $n = k + 1$. Define $v_k : Q_1 \to \mathbb{R}^d$ such that

$$v_k(x, t) := \frac{u(\rho^k x, \rho^{2k} t) - \ell_k(\rho^k x, \rho^{2k} t)}{\rho^{k(1+\alpha)}}.$$

Notice that v_k solves

$$(v_k)_t - \Delta v_k - (p-2) \left\langle D^2 v_k \frac{Dv_k + \rho^{-k\alpha} b_k}{|Dv_k + \rho^{-k\alpha} b_k|}, \frac{Dv_k + \rho^{-k\alpha} b_k}{|Dv_k + \rho^{-k\alpha} b_k|} \right\rangle = f_k \text{in} Q_1,$$

where $f_k := \frac{1}{\rho^{k(\alpha-1)}} f$

Sketch of the proof

Assume that the case $n = k$ has been verified. We prove the case $n = k + 1$. Define $v_k : Q_1 \to \mathbb{R}^d$ such that

$$v_k(x,t) := \frac{u(\rho^k x, \rho^{2k} t) - \ell_k(\rho^k x, \rho^{2k} t)}{\rho^{k(1+\alpha)}}.$$

Notice that v_k solves

$$(v_k)_t - \Delta v_k - (p-2)\left\langle D^2 v_k \frac{Dv_k + \rho^{-k\alpha} b_k}{|Dv_k + \rho^{-k\alpha} b_k|}, \frac{Dv_k + \rho^{-k\alpha} b_k}{|Dv_k + \rho^{-k\alpha} b_k|} \right\rangle = f_k \text{in} Q_1,$$

where $f_k := \frac{1}{\rho^{k(\alpha-1)}} f$.

It follows from the induction hypothesis that v_k satisfies Approximation Lemma, that is, there exists $\tilde{h} \in \mathcal{C}^{2,1}(Q_{7/9})$, so that

$$\sup_{Q_{7/9}} |v_k(x,t) - \tilde{h}(x,t)| < \delta.$$

Sketch of the proof

As a consequence of case $n = 1$, there exists an affine function $\tilde{\ell}$ such that

$$\sup_{Q_\rho} |v_k(x,t) - \tilde{\ell}(x,t)| \leq \rho^{1+\alpha}$$

Sketch of the proof

As a consequence of case $n = 1$, there exists an affine function $\tilde{\ell}$ such that

$$\sup_{Q_\rho} |v_k(x,t) - \tilde{\ell}(x,t)| \leq \rho^{1+\alpha}.$$

Defining $\ell_{k+1}(x,t) := \ell_k(x,t) + \rho^{k(1+\alpha)}\tilde{\ell}(\rho^{-k}x, \rho^{-2k}t)$ yields

$$\sup_{Q_{\rho^{k+1}}} |u(x,t) - \ell_{k+1}(x,t)| \leq \rho^{(k+1)(1+\alpha)}$$

Sketch of the proof

As a consequence of case $n = 1$, there exists an affine function $\tilde{\ell}$ such that

$$\sup_{Q_\rho} |v_k(x,t) - \tilde{\ell}(x,t)| \le \rho^{1+\alpha}.$$

Defining $\ell_{k+1}(x,t) := \ell_k(x,t) + \rho^{k(1+\alpha)}\tilde{\ell}(\rho^{-k}x, \rho^{-2k}t)$ yields

$$\sup_{Q_{\rho^{k+1}}} |u(x,t) - \ell_{k+1}(x,t)| \le \rho^{(k+1)(1+\alpha)}.$$

Also, the coefficients satisfying

$$|a_{k+1} - a_k| \le C\rho^{k(1+\alpha)} \tag{1}$$

and

$$|b_{k+1} - b_k| \le C\rho^{k\alpha} \tag{2}$$

for every $k \in \mathbb{N}$.

Regularity in Hölder Spaces with respect to the time

Theorem (A. -Santos) Let u be a normalized viscosity solution to

$$u_t - \Delta u - (p-2)\left\langle D^2 u \frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|} \right\rangle = f \text{ in } Q_1,$$

where $\xi \in \mathbb{R}^d$ and $f \in L^\infty(Q_1) \cap \mathcal{C}(Q_1)$. If

$$\|f\|_{L^\infty(Q_1)} + |p-2| < \varepsilon,$$

then there exists $C > 0$ such that for all $t \in (-r^2, 0)$

$$|u(0,t) - u(0,0)| \leq C|t|^{\frac{1+\alpha}{2}},$$

for all $\alpha \in (0,1)$.

Ideas behind the proof and some consequences

The result follows from the two main ingredient:

1. Hölder regularity result for the spatial variable
   Andrade-Santos)

Ideas behind the proof and some consequences

The result follows from the two main ingredient:

1. Hölder regularity result for the spatial variable Andrade-Santos);
2. Control the oscillation for a solution of a uniformly parabolic equation

Ideas behind the proof and some consequences

The result follows from the two main ingredient:

1. Hölder regularity result for the spatial variable Andrade-Santos);
2. Control the oscillation for a solution of a uniformly parabolic equation.

Consequences to regularity theory:

1. The solutions with respect the time are of class $\mathcal{C}^{\alpha}$.

Sketch of the proof

For $(x, t) \in Q_r$, we define

$$v(x,t) := u(x,t) - u(0,0) - Du(0,0) \cdot x$$

Sketch of the proof

For $(x,t) \in Q_r$, we define

$$v(x,t) := u(x,t) - u(0,0) - Du(0,0) \cdot x.$$

From the Hölder regularity result, we have

$$|v(x_1,t) - v(x_2,t)| \leq Cr^{1+\alpha},$$

for $x_1, x_2 \in B_r,\ t \in [-r^2, 0]$. Then $\operatorname{osc}_{B_r} v(\cdot,t) \leq Cr^{1+\alpha}$

Sketch of the proof

For $(x,t) \in Q_r$, we define

$$v(x,t) := u(x,t) - u(0,0) - Du(0,0) \cdot x.$$

From the Hölder regularity result, we have

$$|v(x_1,t) - v(x_2,t)| \leq Cr^{1+\alpha},$$

for $x_1, x_2 \in B_r,\ t \in [-r^2, 0]$. Then $\operatorname{osc}_{B_r} v(\cdot,t) \leq Cr^{1+\alpha}$.

Notice that v solves

$$\partial_t v - \Delta v - (p-2)\left\langle D^2 v \frac{Dv+b}{|Dv+b|}, \frac{Dv+b}{|Dv+b|} \right\rangle = f \quad \text{in} \quad Q_r,$$

where $b := Du(0,0)$

Sketch of the proof

For $(x, t) \in Q_r$, we define

$$v(x,t) := u(x,t) - u(0,0) - Du(0,0) \cdot x.$$

From the Hölder regularity result, we have

$$|v(x_1, t) - v(x_2, t)| \leq Cr^{1+\alpha},$$

for $x_1, x_2 \in B_r$, $t \in [-r^2, 0]$. Then $\operatorname{osc}_{B_r} v(\cdot, t) \leq Cr^{1+\alpha}$.

Notice that v solves

$$\partial_t v - \Delta v - (p-2)\left\langle D^2 v \frac{Dv + b}{|Dv + b|}, \frac{Dv + b}{|Dv + b|} \right\rangle = f \quad \text{in} \quad Q_r,$$

where $b := Du(0,0)$. Therefore, we can estimate the oscillation of v in Q_r by $Cr^{1+\alpha} + 4r^2 \|f\|_{L^\infty(Q_1)}$

Sketch of the proof

For $(x,t) \in Q_r$, we define

$$v(x,t) := u(x,t) - u(0,0) - Du(0,0) \cdot x.$$

From the Hölder regularity result, we have

$$|v(x_1,t) - v(x_2,t)| \le Cr^{1+\alpha},$$

for $x_1, x_2 \in B_r,\ t \in [-r^2, 0]$. Then $\operatorname{osc}_{B_r} v(\cdot, t) \le Cr^{1+\alpha}$.

Notice that v solves

$$\partial_t v - \Delta v - (p-2)\left\langle D^2 v \frac{Dv+b}{|Dv+b|}, \frac{Dv+b}{|Dv+b|} \right\rangle = f \quad \text{in} \quad Q_r,$$

where $b := Du(0,0)$. Therefore, we can estimate the oscillation of v in Q_r by $Cr^{1+\alpha} + 4r^2\|f\|_{L^\infty(Q_1)}$.

In particular,

$$|u(0,t) - u(0,0)| = |v(0,t)| \le C|t|^{\frac{1+\alpha}{2}}.$$

Regularity in Sobolev Spaces

Let $u \in \mathcal{C}(Q_1)$ be a viscosity solution to

$$u_t(x,t) - \Delta_p^N u(x,t) = 0$$

Let $u \in \mathcal{C}(Q_1)$ be a viscosity solution to

$$u_t(x,t) - \Delta_p^N u(x,t) = 0.$$

We consider the following regularized problem:

$$\begin{cases} (v_\varepsilon)_t - \Delta v_\varepsilon - (p-2)\dfrac{D^2 v_\varepsilon D v_\varepsilon \cdot D v_\varepsilon}{|Dv_\varepsilon|^2 + \varepsilon^2} & = & 0 & \text{on} & Q_{3/4} \\ v_\varepsilon & = & u & \text{on} & \partial Q_{3/4} \end{cases}$$

Regularity in Sobolev Spaces

Let $u \in \mathcal{C}(Q_1)$ be a viscosity solution to

$$u_t(x,t) - \Delta_p^N u(x,t) = 0.$$

We consider the following regularized problem:

$$\begin{cases} (v_\varepsilon)_t - \Delta v_\varepsilon - (p-2)\dfrac{D^2 v_\varepsilon D v_\varepsilon \cdot D v_\varepsilon}{|Dv_\varepsilon|^2 + \varepsilon^2} & = & 0 & \text{on} & Q_{3/4} \\ v_\varepsilon & = & u & \text{on} & \partial Q_{3/4}. \end{cases}$$

Notice that

1. The solution v_ε is a classical solution (in the interior)

Regularity in Sobolev Spaces

Let $u \in \mathcal{C}(Q_1)$ be a viscosity solution to

$$u_t(x,t) - \Delta_p^N u(x,t) = 0.$$

We consider the following regularized problem:

$$\begin{cases} (v_\varepsilon)_t - \Delta v_\varepsilon - (p-2)\dfrac{D^2 v_\varepsilon D v_\varepsilon \cdot D v_\varepsilon}{|Dv_\varepsilon|^2 + \varepsilon^2} & = & 0 & \text{on} & Q_{3/4} \\ v_\varepsilon & = & u & \text{on} & \partial Q_{3/4}. \end{cases}$$

Notice that

1. The solution v_ε is a classical solution (in the interior);
2. The gradient of v_ε is bounded from above with a bound independent of ε.

Theorem (A. - Santos) Let $u \in \mathcal{C}(Q_1)$ be a viscosity solution to

$$u_t(x,t) - \Delta_p^N u(x,t) = 0$$

Regularity in Sobolev Spaces

Theorem (A. - Santos) Let $u \in \mathcal{C}(Q_1)$ be a viscosity solution to

$$u_t(x,t) - \Delta_p^N u(x,t) = 0,$$

There exists $\varepsilon_1 > 0$ such that if

$$|p-2| \leq \varepsilon_1,$$

then $u \in W^{2,1;q}(Q_{1/2})$ for every $1 < q < \infty$. In addition, there exists a universal constant $C > 0$ such that

$$\|u\|_{W^{2,1;q}(Q_{1/2})} \leq C\|u\|_{L^\infty(Q_1)}.$$

Ideas behind the proof and some consequences

The result follows from two main ingredients:

1. Properties from the regularized equation (K. Does)

Ideas behind the proof and some consequences

The result follows from two main ingredients:

1. Properties from the regularized equation (K. Does);
2. Regularity estimates for uniformly parabolic equations in Sobolev spaces (Lihe Wang)

Ideas behind the proof and some consequences

The result follows from two main ingredients:

1. Properties from the regularized equation (K. Does);
2. Regularity estimates for uniformly parabolic equations in Sobolev spaces (Lihe Wang) .

Consequences to regularity theory:

1. By Sobolev embbeding, we obtain $\mathcal{C}^{1,1^-}$ for the homogeneous case

Ideas behind the proof and some consequences

The result follows from two main ingredients:

1. Properties from the regularized equation (K. Does);
2. Regularity estimates for uniformly parabolic equations in Sobolev spaces (Lihe Wang) .

Consequences to regularity theory:

1. By Sobolev embbeding, we obtain $\mathcal{C}^{1,1^-}$ for the homogeneous case;
2. In our case, the Sobolev estimates implies the stronger version of the Hölder regularity result for $f = 0$;
3. We obtain higher integrability for the viscosity solutions of the problem, with the trade off of losing the precise range on the values of p for which the estimate holds true.

Sketch of the proof

Consider the operator

$$F(D^2u, x, t) := -\Delta u - (p-2)\frac{D^2 u D v_\varepsilon \cdot D v_\varepsilon}{|Dv_\varepsilon|^2 + \varepsilon^2}$$

Sketch of the proof

Consider the operator

$$F(D^2u, x, t) := -\Delta u - (p-2)\frac{D^2uDv_\varepsilon \cdot Dv_\varepsilon}{|Dv_\varepsilon|^2 + \varepsilon^2}.$$

1. The operator F is uniformly parabolic with constants $\lambda = \min(1, p-1)$ and $\Lambda = \max(1, p-1)$

Sketch of the proof

Consider the operator

$$F(D^2u, x, t) := -\Delta u - (p-2)\frac{D^2uDv_\varepsilon \cdot Dv_\varepsilon}{|Dv_\varepsilon|^2 + \varepsilon^2}.$$

1. The operator F is uniformly parabolic with constants $\lambda = \min(1, p-1)$ and $\Lambda = \max(1, p-1)$;
2. The oscillation of F is such that

$$\theta_F(x, t) \leq 2|p-2|$$

Sketch of the proof

Consider the operator

$$F(D^2 u, x, t) := -\Delta u - (p-2)\frac{D^2 u Dv_\varepsilon \cdot Dv_\varepsilon}{|Dv_\varepsilon|^2 + \varepsilon^2}.$$

1. The operator F is uniformly parabolic with constants $\lambda = \min(1, p-1)$ and $\Lambda = \max(1, p-1)$;
2. The oscillation of F is such that

$$\theta_F(x,t) \leq 2|p-2|;$$

3. The function v_ε solves

$$(v_\varepsilon)_t + F(D^2 v_\varepsilon, x, t) = 0.$$

Hence, there exists ε_0 such that if

$$|p - 2| \leq \varepsilon_0$$

Hence, there exists ε_0 such that if

$$|p-2| \leq \varepsilon_0,$$

we obtain that $v_\varepsilon \in W^{2,1;q}(Q_{1/2})$, for all $q \geq 1$ with estimate

$$\|v_\varepsilon\|_{W^{2,1;q}} \leq C.$$

Hence,

1. There exists v_∞ such that $v_\varepsilon \to v_\infty$ uniformly in compact sets;
2. By stability result, we obtain that v_∞ solves the homogeneous normalized p-Laplace equation;
3. It follows from the uniqueness that $v_\infty = u$;
4. Therefore $u \in W^{2,1;q}(Q_{1/2})$ with estimate.

Muito obrigada