On the singular Q-curvature problem

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Conformal geometry and PDE's

- Let (M, g) be a Riemannian manifold. We say that a metric $\tilde{g}$ is conformal to g when there exists a positive function ρ, such that

$$\tilde{g} = \rho g. \qquad \text{(angle preserving)}$$

Conformal geometry

Study of quantities or operators that are conformally invariant.

Conformal geometry and PDE's

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Conformal geometry

Study of quantities or operators that are conformally invariant.

- This connection between conformal geometry and PDE's was explored by H.Yamabe in 1960;

Overview

1. The Yamabe problem
2. The Q-curvature problem
3. Singular setting
4. Strategy
5. Interior analysis
6. Exterior analysis

The Yamabe problem

- In 1960, H. Yamabe proposed the following problem:

Yamabe problem

Given a compact Riemannian manifold (M, g), find a conformal metric $\tilde{g}$ to g with constant scalar curvature.

- If we write $\tilde{g} = u^{\frac{4}{n-2}} g$ $(n \geq 3)$, this problem is equivalent to find a positive solution of the following PDE:

$$\Delta_g u - \frac{n-2}{4(n-1)} R_g u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad M$$

- $L_g := \Delta_g - \frac{n-2}{4(n-1)} R_g$ is called *conformal laplacian*, and it satisfies the property

$$\tilde{g} = u^{\frac{4}{n-2}} g \Rightarrow L_{\tilde{g}}(v) = u^{-\frac{n+2}{n-2}} L_g(uv)$$

Some problems in conformal geometry $\Leftrightarrow$ PDE with critical exponent

- Solved by Yamabe '60, Trudinger '68, Aubin '76 and Schoen '84;
- One of the first PDE with critical exponent to be fully solved;

The Paneitz operator

- In 1983 S. Paneitz discovered a fourth-order operator that we denote by P_g, which has a conformal structure (T. Branson in 1985);
- If $\tilde{g} = u^{\frac{4}{n-4}} g$, then we have the following transformation rule

$$P_{\tilde{g}}(v) = u^{-\frac{n+4}{n-4}} P_g(uv).$$

- This operator is related to a "new" geometric quantity called Q-curvature;

Q-curvature (Some Geometric aspects)

In dimension 4, as an application of the Chern-Gauss-Bonnet Theorem we can verify that

$$4\pi^2\chi(M) = \frac{1}{8}\int_M |W_g|^2 d\mu + \int_M Q_g d\mu$$

- Analogy between the quantity Q_g of a four-manifold and the Gauss curvature of a surface;
    - S.-Y. Alice Chang, M. Eastwood, B. Ørsted and P. C. Yang: *What is Q-curvature?*

Q-curvature problem

Given a (M, g) compact Riemannian manifold, there exists a metric $\tilde{g}$ which is conformal to g with constant Q-curvature?

- Case $n = 4$:
  - Chang-Yang '95 (*Annals*), Djadli-Malchiodi'08 (*Annals*) and Li-Li-Liu'12 (*Adv. Math.*);
- If $n \geq 5$ and $\tilde{g} = u^{\frac{4}{n-4}} g$, then the problem is equivalent to find a positive solution to the following PDE

$$P_g u = \frac{n(n-4)(n^2-4)}{16} u^{\frac{n+4}{n-4}} \quad \text{on} \quad M \tag{1}$$

Let's focus on the case $n \geq 5$;

The Paneitz operator is

$$P_g = \Delta_g^2 + \operatorname{div}\left(\tfrac{4}{n-2}\operatorname{Ric}_g - \tfrac{(n-2)^2+4}{2(n-1)(n-2)} R_g g\right) \mathrm{d} + \tfrac{n-4}{2} Q_g .$$

and the Q-curvature is given by

$$Q_g = -\tfrac{1}{2(n-1)} \Delta_g R_g + \tfrac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2} R_g^2 - \tfrac{2}{(n-2)^2} |\operatorname{Ric}_g|^2,$$

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What are the main difficulties?

- Variational problem, but ...
- Lack of maximum principle!
  - Several new insights thanks to the works of Qing-Raske' 06 *IMRN*, Gursky- Malchiodi '15 *JEMS* and Hang-Yang'16 CPAM;

What happens in the noncompact case?

- What happens if the domain is noncompact? Ex: $M\backslash X$, $X \subset M$ is a nonempty set.

Problem

Given a compact manifold (M, g), find a conformal metric $\tilde{g}$ which is complete in $M\backslash X$ with constant scalar (Q-curvature) curvature .

- *Singular Yamabe problem (SYP)* and *Singular Q-curvature problem*;

In the case of the SYP for $n \geq 3$, if u is a positive solution of

$$\begin{cases} \Delta_g u - \frac{n-2}{4(n-1)} R_g u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 & \text{in} \quad M \backslash X \\ u(x) \to \infty & \text{as} \quad x \to X \end{cases}$$

then the metric $\tilde{g} = u^{\frac{4}{n-2}} g$ is complete and has constant positive scalar curvature.

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Existence of solutions is directly related with the size of the singular set X and the sign of the scalar curvature!

- Aviles and McOwen' 88: If X is a k-submanifold, then a solution for SYP with $R_g < 0$ exists iff, $k > (n-2)/2$.
- If a solution with $R_g \geq 0$ exists then $\dim X \leq (n-2)/2$;

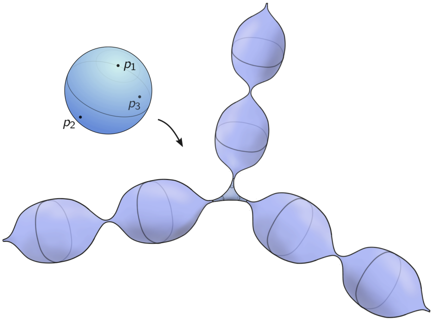
Existence results: Case $R_g \geq 0$

- Mazzeo, Pacard '96 *JDG*:
  - X closed submanifold of M, $0 < dimX \leq (n-2)/2$;

What if $X = \{p_1, \ldots, p_k\}$? $(R_g = n(n-1))$;

- Mazzeo,Pacard '99 *Duke*, R. Schoen '88 *CPAM*:
  - $M = \mathbb{S}^n$, X has at least two points;
- Byde '05 *Indiana*:
  - M is conformally flat and g is nondegenerate;
- A.Silva Santos '09 *Ann.Henri Poincaré*;
  - g is nondegenerate and Weyl tensor vanishes at the singular points;

Interesting facts...



- Analogy between Constant Scalar curvature metrics and CMCs surfaces ;
- Different types of problems: extrinsic and intrisic;

Singular Q-curvature problem

Theorem (Hyder-Sire '20 *JFA*)

Let X be a connected closed submanifold of M. Assume that $\boldsymbol{Q_g \geq 0}$, $Q_g \not\equiv 0$ and $\boldsymbol{R_g \geq 0}$. If $0 < dim(X) \leq (n-4)/2$, then there exists an infinite dimensional family of complete metrics on $M \setminus X$ with constant Q-curvature.

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What happens when the singular set $X = \{p_1, ..., p_k\}$?

Main result

- Suppose $X = \{p\}$;

Theorem

Let (M^n, g) be a closed Riemannian manifold. Assume that $Q_g = n(n^2-4)/8$ and that g satisfies (H1) and (H2). Then, there exist a constant $\varepsilon_0 > 0$ and a one-parameter family of metrics $\{g_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}$, conformal to g in $M \backslash \{p\}$ with constant Q-curvature.

General Idea: PDE viewpoint

If we write

$$u = u_0 + v$$

where u_0 is a good approximation for the solution, we have

$$0 = H_g(u_0 + v) = H_g(u_0) + L_g^{u_0}(v) + Q_0(v) \Rightarrow$$

$$L_g^{u_0}(v) = -H_g(u_0) - Q_0(v)$$

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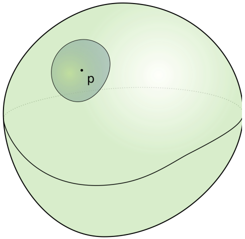
If the linearized operator has a right inverse G_g^0, then find a solution of the problem is equivalent to find a fixed point of the operator

$$N(v) = G_g^0(-H_g(u_0) - Q_0(v)).$$

Q-curvature case

Gluing method:

1. Interior analysis (near the singularity)
2. Exterior analysis
3. Gluing



- The asymptotic analysis by J. Ratzkin '20 indicates that the Delaunay solutions are good approximations near the singularity;

Local models: Delaunay solutions

Consider the positive solutions $u > 0$ of the fourth-order problem

$$\Delta^2 u = \frac{n(n-4)(n^2-4)}{16} u^{\frac{n+4}{n-4}} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\} \tag{D}$$

which are singular at the origin.

- C-S. Lin '98 *CMH*: Positive solutions of (D) are radially symmetric.
- R.L. Frank and T. König '20 classified all the solutions of (D);

Theorem (Frank, König '20 *APDE*)

If the origin is a nonremovable singularity. Then there exists $\varepsilon \in (0, \varepsilon_0)$ *and* $T \in (0, T_\varepsilon)$, *such that*

$$u_{\varepsilon,T}(x) = |x|^{\frac{4-n}{2}} v_\varepsilon(-\ln|x| + T).$$

The solutions of (D) are called Delaunay solutions.

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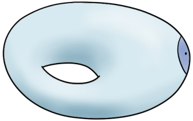
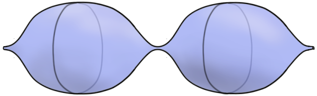
- Even in fourth-order, this solutions are depending on two-parameters: $\boldsymbol{\varepsilon}$ (necksize), and $\boldsymbol{T}$ (period);

General idea: Geometric viewpoint





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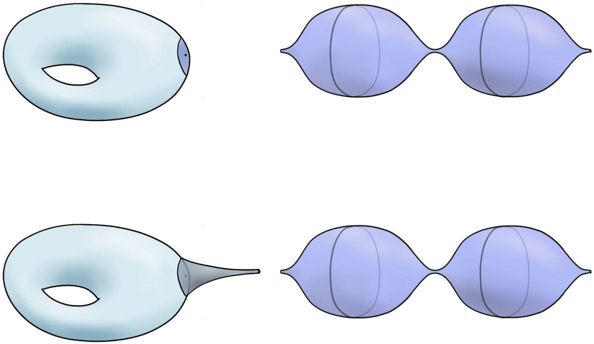

- Constant function 1 + Green function with pole at p;

Interior analysis

Let $a \in \mathbb{R}^n$ and $R \in \mathbb{R}$. Consider the modified Delaunay solutions $u_{\varepsilon,R,a}$.

Consider the operator

$$H_g(u) = P_g u - \frac{n(n-4)(n^2-4)}{16} u^{\frac{n+4}{n-4}}$$

We will seek solutions of the type $u_{\varepsilon,R,a} + v > 0$ such that

$$\begin{cases} H_g(u_{\varepsilon,R,a} + v) = 0 & \text{in} \quad B_r(p)\backslash\{p\} \\ (u_{\varepsilon,R,a} + v)(x) \to \infty & \text{as} \quad x \to p \end{cases}$$

Expanding this equation, we obtain

$$H_g(u_{\varepsilon,R,a} + v) = H_g(u_{\varepsilon,R,a}) + L_g^{u_{\varepsilon,R,a}}(v) + Q_{\varepsilon,R,a} = 0$$

which is equivalent to solve

$$L_\delta^{u_{\varepsilon,R,a}} v = -H_g(u_{\varepsilon,R,a}) - L_g^{u_{\varepsilon,R,a}}(v) + L_\delta^{u_{\varepsilon,R,a}}(v) - Q_{\varepsilon,R,a}$$

Understand the linearized operator, denoted by $L_{\varepsilon,R,a} := L_\delta^{u_{\varepsilon,R,a}}$. Is there any "good space" in which this operator admits a right inverse?

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Understand the linearized operator, denoted by $L_{\varepsilon,R,a} := L_\delta^{u_{\varepsilon,R,a}}$. Is there any "good space" in which this operator admits a right inverse?

- Weighted Hölder spaces;

Linearized operator

Proposition

Let $R > 0$, $\alpha \in (0,1)$ and $\mu \in (1,2)$. Then there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, $a \in \mathbb{R}^n$ and $0 < r < 1$ with $|a| r \leq r_0$ for some $r_0 \in (0,1)$, there is an operator

$$G_{\varepsilon,R,a,r} : C^{0,\alpha}_{\mu-4}(B_r(0)\backslash\{0\}) \to C^{4,\alpha}_{\mu}(B_r(0)\backslash\{0\})$$

with the norm bounded independently of ε, R, a and r, which is the right inverse of $L_{\varepsilon,R,a}$.

- This operator prescribes the Navier condition on the boundary: (high eingemodes);
  - $\lambda_0, \lambda_1, \ldots, \lambda_n, \lambda_{n+1}, \ldots$, eingenvalues of $\Delta_{\mathbb{S}^n}$;

Model operator

How can we add a term that "controls" what happens on the boundary?

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Proposition

Let $0 < \alpha < 1$. There exists a bounded linear operator $\mathcal{P}_r : \pi_r''(C^{4,\alpha}(\mathbb{S}_r^{n-1})) \times C^{4,\alpha}(\mathbb{S}_r^{n-1}) \to C_2^{4,\alpha}(B_r(0) \setminus \{0\})$, such that for all $(\phi_0, \phi_2) \in C^{4,\alpha}(\mathbb{S}^{n-1}, \mathbb{R}^2)$, it holds

$$\begin{cases} \Delta^2 \mathcal{P}_r(\phi_0, \phi_2) & = & 0 & \text{in} & B_r(0) \setminus \{0\} \\ \Delta \mathcal{P}_r(\phi_0, \phi_2) & = & r^{-2} \phi_2 & \text{on} & \partial B_r \\ \pi''(\mathcal{P}_r(\phi_0, \phi_2)) & = & \phi_0 & \text{on} & \partial B_r. \end{cases}$$

Interior analysis

Finaly, we will try to solve

$$H_g(u_{\varepsilon,R,a} + v_{\phi_0,\phi_2} + v) = 0$$

which is equivalent to

$$L_{\varepsilon,R,a}(v) = \textit{the rest of the expansion}.$$

We need to show that the RHS belongs to the domain of the right inverse.

- For dimensions $5 \leq n \leq 7$ that it is true!
- For $n \geq 8$, we need to assume that

$$\nabla^l W_g(p) = 0, \quad \text{for } l = 0, 1, \ldots, \left[\frac{n-8}{2}\right]. \tag{H1}$$

- Related with the *Weyl vanishing conjecture*;
- (H1) implies that the metric has a good decay; (Some technical difficulties are overcome by the construction of an auxiliary function)

What about the exterior domain?

Exterior analysis

- 1 is an approx. solution + Green function $G_p(x) \approx |x|^{4-n}$ of the linearized;
- We assume that

$$g \text{ is nondegenerate} \tag{H2}$$

Exterior analysis

- 1 is an approx. solution + Green function $G_p(x) \approx |x|^{4-n}$ of the linearized;
- We assume that

$$g \text{ is nondegenerate} \tag{H2}$$

  - Green function G_p exists;
  - Linearized operator is invertible;

Therefore,

$$H_g(1 + \lambda G_p + v) = 0 \qquad \text{in} \qquad M \backslash B_r(p)$$

which is equivalent to

$$L_g^1(v) = -Q^1(\lambda G_p + v)$$

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which is equivalent to

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What about the boundary?

Exterior Poisson operator

Proposition

Let $0 < \alpha < 1$. There exists a bounded linear operator $\mathcal{Q}_r : C^{4,\alpha}(\mathbb{S}_r^{n-1}, \mathbb{R}^2) \to C^{4,\alpha}_{4-n}(\mathbb{R}^n \backslash B_r)$ such that for all $(\psi_0, \psi_2) \in C^{4,\alpha}(\mathbb{S}_r^{n-1}, \mathbb{R}^2)$, it holds

$$\begin{cases} \Delta^2 \mathcal{Q}_r(\psi_0, \psi_2) & = & 0 & \text{in} & \mathbb{R}^n \backslash B_r \\ \Delta \mathcal{Q}_r(\psi_0, \psi_2) & = & r^{-2}\psi_2 & \text{on} & \partial B_r \\ \mathcal{Q}_r(\psi_0, \psi_2) & = & \psi_0 & \text{on} & \partial B_r . \end{cases} \tag{2}$$

Gluing Procedure

Interior solution	Exterior solution
$u_\varepsilon := u_\varepsilon(a, R, \varphi_0, \varphi_2)$	$v_\varepsilon := v_\varepsilon(\lambda, \psi_0, \psi_2)$

To prove that the solutions will "glue", we need to show that

$$\begin{cases} u_\varepsilon & = & v_\varepsilon \\ \partial_r u_\varepsilon & = & \partial_r v_\varepsilon \\ \Delta_g u_\varepsilon & = & \Delta_g v_\varepsilon \\ \partial_r \Delta_g u_\varepsilon & = & \partial_r \Delta_g v_\varepsilon \end{cases} \tag{$\mathcal{G}$}$$

- To show the existence of parameters satisfying $(\mathcal{G})$, we use fixed points arguments again;
- Spectral decomposition: low and high eigenmodes;

Theorem (Main result):

Let (M^n, g) be a closed Riemannian manifold, $n \geq 5$. Assume that $Q_g = n(n^2-4)/8$ and

(H1) The Weyl tensor vanishes at p, up to order $[(n-8)/2]$ $(n \geq 8)$;

(H2) g is nondegenerate;

Then, there exist a constant $\varepsilon_0 > 0$ and a one-parameter family of metrics $\{g_\varepsilon\}$, conformal to g in $M \backslash \{p\}$ with constant Q-curvature. Moreover, each g_ε is asymptotically Delaunay and $g_\varepsilon \to g$ uniformly on compact sets as $\varepsilon \to 0$.

Thank you!

Modified Delaunay solutions

$$u_{\varepsilon,R,a}(x) = |x - a|x|^2|^{\frac{4-n}{2}} v_\varepsilon \left(-\log|x| + \log\left|\frac{x}{|x|} - a|x|\right| + \log R \right).$$

Definition

A metric g is **nondegenerate** if the linearized operator $L_g : C^{4,\alpha}(M) \to C^{0,\alpha}(M)$ is surjective for some $\alpha \in (0,1)$, where

$$L_g(u) = P_g u - \frac{n(n^2-4)(n+4)}{16} u.$$