Regularity theory for a free boundary problem with double degeneracy

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Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^n$, we investigate the doubly degenerate fully non-linear elliptic problem

$$\begin{cases} \mathcal{H}(x, \nabla u) F(x, D^2 u) & = & f(x) & \text{in} & \Omega_+(u), \\ |\nabla u| & = & \mathrm{Q}(x) & \text{on} & \mathfrak{F}(u). \end{cases} \tag{1}$$

where

- $\mathcal{H} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ behaves as

$$L_1 \cdot \mathcal{K}_{p,q,\mathfrak{a}}(x, |\xi|) \le \mathcal{H}(x, \xi) \le L_2 \cdot \mathcal{K}_{p,q,\mathfrak{a}}(x, |\xi|)$$

for constants $0 < L_1 \le L_2 < \infty$, with

$$\mathcal{K}_{p,q,\mathfrak{a}}(x, |\xi|) := |\xi|^p + \mathfrak{a}(x)|\xi|^q, \ \text{ for } \ (x, \xi) \in \Omega \times \mathbb{R}^n;$$

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- $F : \Omega \times \mathrm{Sym}(n) \to \mathbb{R}$ is a uniformly elliptic operator;

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- $u \geq 0$ in Ω;
- $\Omega^+(u) := \{x \in \Omega : u(x) > 0\}$;
- $\mathfrak{F}(u) := \partial\Omega^+(u) \cap \Omega$.

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- In a first moment, we establish optimal Lipschitz regularity to viscosity solutions;
- We prove the non-degeneracy of solutions;
- The next step is to investigate the regularity of the free boundary $\mathfrak{F}(u)$. We prove that flat free boundaries are of class $\mathcal{C}^{1,\beta}$;
- Finally, we prove that Lipschitz free boundaries are os class $\mathcal{C}^{1,\beta}$.

Motivation

One of the main signatures of this model is its interplay between two different kinds of degeneracy laws, in accordance with the values of the modulating function $\mathfrak{a}$.

$$\Omega \times \mathbb{R}^n \ni (x,\xi) \mapsto \mathcal{H}(x,\xi) \propto |\xi|^p + \mathfrak{a}(x)|\xi|^q \quad 0 < p < q < \infty$$

$$\text{and} \quad 0 \leq \mathfrak{a} \in C^0(\Omega).$$

Motivation

For related regularity estimates and free boundary problems driven by second order operators with a single degeneracy law:

- Araújo, Ricarte, Teixeira - (Calc. Var. PDE - 2015), (Ann. Inst. H. Poincaré Anal. Non Linéaire - 2017);
- Birindelli, Demengel - (ESAIM Control Optim. Calc. Var. - 2014);
- Birindelli, Demengel, Leoni - (NoDEA - 2019);
- Da Silva, Leitão, Ricarte - (Math. Nachr. - 2021);
- Da Silva, Vivas - (Rev. Mat. Iberoam. - 2021), (Discrete and Continuous Dynamical Systems, 2021);
- Imbert, Silvestre - (Adv. Math. - 2012), (JEMS - 2016).

Motivation

The model in (1) can be considered the non-divergence form of certain variational integrals from the Calculus of Variations with double phase structure:

$$(w, f) \mapsto \min \int_{\Omega} \left(\frac{1}{p} |\nabla w|^p + \frac{\mathfrak{a}(x)}{q} |\nabla w|^q - fw \right) dx,$$

where $(w, f) \in \left(W_0^{1,p}(\Omega) + u_0, L^m(\Omega) \right)$, $\mathfrak{a} \in C^{0,\alpha}(\Omega, [0, \infty))$, for some $0 < \alpha \leq 1$, $1 < p \leq q < \infty$ and $m \in (n, \infty]$;

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- Colombo, Mingione - (Arch. Rational Mech. Anal. - 2015);
- Baroni, Colombo, Mingione - (Calc. Var. PDE - 2018);
- De Filippis - (JDE - 2019);
- De Filippis, Mingione - (The Journal of Geometric Analysis - 2020).

Motivation

Historically the mathematical investigation of the regularity of the free boundary $\mathfrak{F}(u)$ in problems like (1) has a large literature and it has presented wide advances in the last three decades or so.

Motivation

Uniform Elliptic case - Variational approach.

The case $f = 0$ and $\mathcal{H}(x, \xi) = 1$, was studied by minimizing

$$\mathcal{J}(u) := \int_{\Omega \cap \{u>0\}} \mathfrak{f}(x, u(x), \nabla u(x)) dx \longrightarrow \min,$$

where was proved existence of a minimum and regularity of the free boundary via blow-up techniques, or via singular perturbation methods for the problem $\Delta u_\varepsilon = \beta_\varepsilon(u_\varepsilon)$.

- Alt, Caffarelli - (J. Reine Angew. Math. - 1981);
- Caffarelli - (Rev. Mat. Iber. - 1987), (Comm. Pure Appl. Math. - 1989).

Motivation

Degenerate cases – Variational approach.

- Danielli, Petrosyan - (Calc. Var. PDE - 2005) established the regularity near "flat points" of the free boundary of non-negative solutions to the minimization problem

$$\min \mathcal{J}_p(u) \qquad \text{with} \qquad \mathcal{J}_p(u) := \int_\Omega \left(|\nabla u|^p + \lambda_0^p \chi_{\{u>0\}} \right) dx,$$

which is governed by the p-Laplacian operator, for $f = 0$, $1 < p < \infty$ and $\lambda > 0$.

Motivation

- Martínez, Wolanski - (Adv. Math. - 2008) study the optimization problem of minimizing

$$\min \mathcal{J}_{\mathrm{G}}(u) \qquad \text{with} \qquad \mathcal{J}_{\mathrm{G}}(u) := \int_{\Omega} \left(\mathrm{G}(|\nabla u|) + \lambda_0 \chi_{\{u>0\}}\right) dx,$$

in an Orlicz-Sobolev scenario, thereby extending the Alt-Caffarelli's theory.

Motivation

The study of existing, Lipschitz regularity and regularity of the free boundary for homogeneous/inhomogeneous free boundary problems driven by $p(x)$-Laplacian type operators as follows

$$\begin{cases} \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) & = & f(x) & \text{in} & \Omega_+(u), \\ |\nabla u| & = & \lambda^*(x) & \text{on} & \mathfrak{F}(u). \end{cases}$$

can be found

- Fernández Bonder, Martínez, Wolanski - (Nonlinear Anal. - 2010;
- Lederman, Wolanski - (Interfaces Free Bound., 2017), (J. Math. Anal. Appl., 2019), (Math. Eng., 2021).

Motivation

Uniform elliptic case – Non-variational approach.

- Feldman - (Indiana Univ. Math. J. - 2001)

For the context of fully non-linear elliptic equations, the homogeneous problem, i.e. $f = 0$ (with $\mathcal{H}(x, \xi) \equiv 1$).

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- De Silva - (Interfaces Free Bound. - 2011);
- De Silva, Ferrari, Salsa - (J. Math. Pures Appl. - 2015).

For the non-homogeneous case, $f \neq 0$ (with $\mathcal{H}(x, \xi) \equiv 1$), in the one and two-phase scenarios, respectively.

Motivation

In turn, the FBP considered in (1) also appears as the limit of certain inhomogeneous singularly perturbed problems in the non-variational context of high energy activation model in combustion and flame propagation theories.

- Araújo, Ricarte, Teixeira - (Ann. Inst. H. Poincaré Anal. Non Linéaire - 2017);
- Ricarte, Silva - (Interfaces and Free Bound. - 2015);
- Ricarte, Teixeira - (J. Funct. Anal. - 2011).

Motivation

The simplest mathematical model (in this case) is given by: for each $\varepsilon > 0$ fixed, we seek a non-negative profile u^ε satisfying

$$\begin{cases} \left[|\nabla u^\varepsilon|^p + \mathfrak{a}(x)|\nabla u^\varepsilon|^q\right] \Delta u^\varepsilon & = & \frac{1}{\varepsilon}\beta\left(\frac{u^\varepsilon}{\varepsilon}\right) + f_\varepsilon(x) & \text{in} & \Omega, \\ u^\varepsilon(x) & = & g(x) & \text{on} & \partial\Omega, \end{cases}$$

in the viscosity sense for suitable data $p, q \in (0, \infty)$, $\mathfrak{a}$, g, where β_ε behaves singularly of order $\mathrm{o}\left(\varepsilon^{-1}\right)$ near ε-level surfaces. In such a scenario, existing solutions are locally (uniformly) Lipschitz continuous.

- da Silva, Júnior, Ricarte - (Rev. Mat. Iberoam. - 2022)

Assumptions

Definition
Let $u \in C(\Omega)$ nonnegative. We say that u is a viscosity supersolution (resp.subsolution) to

$$\begin{cases} \mathcal{H}(x, \nabla u) F(x, D^2 u) & = & f(x) & \text{in} & \Omega_+(u), \\ |\nabla u| & = & \mathrm{Q}(x) & \text{on} & \mathfrak{F}(u). \end{cases}$$

if and only if the following conditions are satisfied:

(F1) *If* $\phi \in C^2(\Omega^+(u))$ *touches* u *by below (resp. above) at* $x_0 \in \Omega^+(u)$ *then*

$$\mathcal{H}(x_0, \nabla\phi(x_0))F(x_0, D^2\phi(x_0)) \le f(x_0)$$

$$\left(resp.\ \mathcal{H}(x_0, \nabla\phi(x_0))F(x_0, D^2\phi(x_0) \ge f(x_0)\right).$$

(F2) *If* $\phi \in C^2(\Omega)$ *and* ϕ *touches* u *below (resp. above) at* $x_0 \in \mathfrak{F}(u)$ *and* $|\nabla\phi|(x_0) \neq 0$ *then*

$$|\nabla\phi|(x_0) \le Q(x_0) \qquad (resp.\ |\nabla\phi|(x_0) \ge Q(x_0)).$$

We say that u *is a viscosity solution if it is a viscosity supersolution and a viscosity subsolution.*

Continuity and normalization condition

We suppose that

$\Omega \ni x \mapsto F(x, \cdot) \in C^0(\mathrm{Sym}(n)) \quad \text{and} \quad F(\cdot, \mathrm{O}_n) = 0$ where O_n is the zero matrix.

This normalizing assumption can be impose without loss of generality.

Uniform ellipticity

For any pair of matrices $\mathrm{X}, \mathrm{Y} \in \mathit{Sym}(n)$

$$\mathcal{P}^{-}_{\lambda,\Lambda}(\mathrm{X}-\mathrm{Y}) \leq F(x,\mathrm{X}) - F(x,\mathrm{Y}) \leq \mathcal{P}^{+}_{\lambda,\Lambda}(\mathrm{X}-\mathrm{Y})$$

where $\mathcal{P}^{\pm}_{\lambda,\Lambda}$ stand for *Pucci's extremal operators* given by

$$\mathcal{P}^{-}_{\lambda,\Lambda}(\mathrm{X}) := \lambda \sum_{e_i>0} e_i(\mathrm{X}) + \Lambda \sum_{e_i<0} e_i(\mathrm{X})$$

$$\text{and} \qquad \mathcal{P}^{+}_{\lambda,\Lambda}(X) := \Lambda \sum_{e_i>0} e_i(\mathrm{X}) + \lambda \sum_{e_i<0} e_i(\mathrm{X})$$

for *ellipticity constants* $0 < \lambda \leq \Lambda < \infty$, where $\{e_i(\mathrm{X})\}_i$ are the eigenvalues of X.

$\omega-$continuity of coefficients

There exist a uniform modulus of continuity $\omega : [0, \infty) \to [0, \infty)$ and a constant $\mathrm{C_F} > 0$ such that

$$\Omega \ni x, x_0 \mapsto \Theta_{\mathrm{F}}(x, x_0) := \sup_{\substack{\mathrm{X} \in Sym(n) \\ \mathrm{X} \neq 0}} \frac{|F(x, \mathrm{X}) - F(x_0, \mathrm{X})|}{\|\mathrm{X}\|} \leq \mathrm{C_F} \omega(|x - x_0|),$$

which measures the oscillation of coefficients of F around x_0. Finally, we define

$$\|F\|_{C^{\omega}(\Omega)} := \inf \left\{ \mathrm{C_F} > 0 : \frac{\Theta_{\mathrm{F}}(x, x_0)}{\omega(|x - x_0|)} \leq \mathrm{C_F}, \ \forall\, x, x_0 \in \Omega, \ x \neq x_0 \right\}.$$

In our studies, the diffusion properties of the model (1) degenerate along an *a priori* unknown set of singular points of existing solutions:

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For this reason, we will enforce that $\mathcal{H} : \Omega \times \mathbb{R}^n \to [0,\infty)$ behaves as

$$L_1 \cdot \mathcal{K}_{p,q,\mathfrak{a}}(x,|\xi|) \le \mathcal{H}(x,\xi) \le L_2 \cdot \mathcal{K}_{p,q,\mathfrak{a}}(x,|\xi|)$$

for constants $0 < L_1 \le L_2 < \infty$, where

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In addition, we suppose that the exponents p, q and the modulating function $\mathfrak{a}(\cdot)$ fulfil

$$0 < p \leq q < \infty \qquad \text{and} \qquad \mathfrak{a} \in C^0(\Omega, [0, \infty)).$$

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$$0 < p \leq q < \infty \qquad \text{and} \qquad \mathfrak{a} \in C^0(\Omega, [0, \infty)).$$

Finally, we will assume the following condition: there exist a universal constant $\mathrm{C}_{\mathfrak{a}} > 0$ and a modulus of continuity $\omega_{\mathfrak{a}} : [0, \infty) \to [0, \infty)$ such that

$$|\mathcal{H}(x, \xi) - \mathcal{H}(y, \xi)| \leq \mathrm{C}_{\mathfrak{a}} \omega_{\mathfrak{a}}(|x - y|)|\xi|^q, \quad \forall \ (x, y, \xi) \in \Omega \times \Omega \times \mathbb{R}^n.$$

Optimal Lipschitz Regularity

Lipschitz regularity of solutions

Theorem (da Silva; R.; Ricarte; Vivas - To appear in Israel J. Math.)
Let $\mathrm{Q} \in C^0(B_1; [0, \infty)) \cap L^\infty(B_1; [0, \infty))$ *and u be a bounded viscosity solution to (1) in* B_1*. Then, there exists a universal constant* $\mathrm{C}_1 = \mathrm{C}_1(n, \lambda, \Lambda, \mathfrak{a}, L_1, p, q) > 0$ *such that*

$$u(x_0) \leq \mathrm{C}_1 . \left(\|u\|_{L^\infty(B_1)} + \|\mathrm{Q}\|_{L^\infty(B_1)} + \max\left\{ \|f\|_{L^\infty(B_1)}^{\frac{1}{p+1}}, \|f\|_{L^\infty(B_1)}^{\frac{1}{q+1}} \right\} \right) \mathit{dist}(x_0, \mathfrak{F}(u)),$$

for all $x_0 \in B_{1/2}$*; i.e., solutions have at most linear behavior close to free boundary points. Particularly, there exists* $C_2 = C_2(n, \lambda, \Lambda, L_1, p, q, \|F\|_{\mathcal{C}^\omega}) > 0$ *such that*

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq \mathrm{C}_2 . \left(\|u\|_{L^\infty(B_1)} + \|\mathrm{Q}\|_{L^\infty(B_1)} + \max\left\{ \|f\|_{L^\infty(B_1)}^{\frac{1}{p+1}}, \|f\|_{L^\infty(B_1)}^{\frac{1}{q+1}} \right\} + 1 \right).$$

The proof of the Lipschitz regularity can be obtained employing some ideas performed for the scenario of singularly perturbed FBPs.

- Araújo, Ricarte, Teixeira - (Ann. Inst. H. Poincaré Anal. Non Linéaire - 2017);
- da Silva, Júnior, Ricarte - (Rev. Mat. Iberoam. - 2022);
- Ricarte, Silva - (Interfaces and Free Bound., 2015);
- Ricarte, Silva, Teymurazyan - (JDE - 2017);
- Ricarte, Teixeira - (J. Funct. Anal. - 2011).

Sketch of the proof:

- Take $x_0 \in B_{1/2}$ such that $x_0 \in B^+_{1/2}(u)$. We suppose that $\mathrm{dist}(x_0, \mathfrak{F}(u)) \leq 1/2$ and consider the scaled funtion

$$v_{x_0,d_0}(x) := \frac{u(x_0 + rd_0x)}{\mathrm{dist}(x_0, \mathfrak{F}(u))}$$

for $0 < r < 1$ to be chosen.

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  for $0 < r < 1$ to be chosen.

- The idea is to prove that $v_{x_0,d_0}(0) \leq C_0$ for some universal constant $C_0 > 0$.

- v_{x_0,d_0} is a non-negative viscosity solution of

$$\mathcal{H}_{x_0,d_0}(x,\nabla v_{x_0,d_0})F_{x_0,d_0}(x,D^2 v_{x_0,d_0}) = f_{x_0,d_0}(x) \quad \text{in} \quad B_1$$

where

$$\begin{cases} F_{x_0,d_0}(x,\mathrm{X}) & := & r^2 d_0 F\left(x_0 + r d_0 x, \frac{1}{r^2 d_0}\mathrm{X}\right) \\ \mathcal{H}_{x_0,d_0}(x,\xi) & := & r^p \mathcal{H}\left(x_0 + r d_0 x, \frac{1}{r}\xi\right) \\ \mathfrak{a}_{x_0,d_0}(x) & := & r^{p-q}\mathfrak{a}(x_0 + r d_0 x) \\ f_{x_0,d_0}(x) & := & r^{p+2} d_0 f(x_0 + r d_0 x) \\ \mathrm{Q}_{x_0,d_0}(x) & := & r\mathrm{Q}(x_0 + r d_0 x) \end{cases}$$

where $F_{x_0,d_0}, \mathcal{H}_{x_0,d_0}$ and $\mathfrak{a}_{x_0,d_0}$ satisfy the structural assumptions.

- Consider the annulus $\mathcal{A}_{\frac{1}{2},1} := B_1 \setminus B_{\frac{1}{2}}$ and the barrier function $\Phi : \overline{\mathcal{A}_{\frac{1}{2},1}} \to \mathbb{R}_+$ given by

$$\Phi(x) = \mu_0 \cdot \left(e^{-\delta |x|^2} - e^{-\delta} \right)$$

where $\mu_0, \delta > 0$ will be chosen.

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$$\Phi(x) = \mu_0 \cdot \left(e^{-\delta |x|^2} - e^{-\delta} \right)$$

  where $\mu_0, \delta > 0$ will be chosen.

- We show that Φ is a strict viscosity subsolution to

$$\mathcal{H}_{x_0,d_0}(x, \nabla\Phi) F_{x_0,d_0}(x, D^2\Phi) = f_{x_0,d_0}(x) \quad \text{in} \quad \mathcal{A}_{\frac{1}{2},1}$$

  provided we may adjust appropriately the values of $\mu_0, \delta > 0$ and $r > 0$.

- Choose $\mu_0 := (e^{-\delta/4} - e^{-\delta})^{-1} \cdot \inf_{\partial B_{\frac{1}{2}}} v_{x_0,d_0}(x) > 0$. It follows that

$$\Phi(x) \leq v_{x_0,d_0}(x) \quad \text{on} \quad \partial \mathcal{A}_{\frac{1}{2},1}.$$

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- Let $z_0 \in \mathfrak{F}(v_{x_0,d_0})$ be a point that achieves the distance, i.e., $rd_0 = |x_0 - z_0|$ and consider $y_0 := \frac{z_0 - x_0}{rd_0} \in \partial B_1$.

- Taking into account the free boundary condition, we obtain concerning the normal derivatives in the direction ν at x_0 the following

$$\mu_0 \delta e^{-\delta} \leq \frac{\partial \Phi(y_0)}{\partial \nu} \leq \|\mathrm{Q}\|_{L^\infty(B_1)},$$

which implies that

$$\inf_{\partial B_{\frac{1}{2}}} v_{x_0,d_0}(x) \leq \|\mathrm{Q}\|_{L^\infty(B_1)} \mathrm{C}(\delta).$$

- By invoking the Harnack inequality and the definition of v_{x_0,d_0}, it follows that

$$\sup_{B_{\frac{rd_0}{2}}(x_0)} u(x) \leq \mathrm{C}_0 d_0 \cdot \left\{ \|\mathrm{Q}\|_{L^\infty(B_1)} + \max\left\{ (r^{p+2} d_0)^{\frac{1}{p+1}}, (r^{p+2} d_0)^{\frac{1}{q+1}} \right\} \Pi_{p,q}^{f,\mathfrak{a}_{x_0,d_0}} \right\}.$$

- By invoking the Harnack inequality and the definition of v_{x_0,d_0}, it follows that

$$\sup_{B_{\frac{rd_0}{2}}(x_0)} u(x) \leq \mathrm{C}_0 d_0 \cdot \left\{ \|\mathrm{Q}\|_{L^\infty(B_1)} + \max\left\{ (r^{p+2} d_0)^{\frac{1}{p+1}}, (r^{p+2} d_0)^{\frac{1}{q+1}} \right\} \Pi_{p,q}^{f,\mathfrak{a}_{x_0,d_0}} \right\}.$$

- By $\mathcal{C}^{1,\beta}_{\text{loc}}$-estimates we have

$$|\nabla u(x_0)| \leq \mathrm{C} \cdot \left(\|v_{x_0,d_0}\|_{L^\infty\left(B_{\frac{1}{2}}\right)} + 1 + \|f_{x_0,d_0}\|^{\frac{1}{p+1}}_{L^\infty\left(B_{\frac{1}{2}}\right)} \right).$$

Non-degeneracy of solutions

Theorem (da Silva; R.; Ricarte; Vivas - To appear in Israel J. Math.)
Let $\mathrm{Q} \in C^0(B_1; [0,\infty)) \cap L^\infty(B_1; [0,\infty))$ *and* u *be a bounded viscosity solution to (1) in* B_1*. Further, suppose that* $\mathfrak{F}(u)$ *is a Lipschitz graph in* B_1 *with* $\mathfrak{F}(u) \cap B^+_{1/2}(u) \neq \emptyset$*. There exists a universal* $\eta_0 \in (0,1)$ *and a universal constant* $\mathrm{C}_* = \mathrm{C}(n, \lambda, \Lambda, p, q, \|F\|_{C^\omega(B_1)}) > 0$ *such that if*

$$\|\mathrm{Q} - 1\|_{L^\infty(B_1)} < \eta_0$$

then

$$u(x_0) \geq \mathrm{C}_*.dist(x_0, \mathfrak{F}(u)),$$

for all $x_0 \in B^+_{1/2}(u)$*; i.e. solutions growth at least in a linear fashion close to free boundary points.*

For the proof of the non-degeneracy result, the argument follows as the one in [De Silva - Interfaces Free Bound., 2011], after constructing the appropriate barrier.

For the proof of the non-degeneracy result, the argument follows as the one in [De Silva - Interfaces Free Bound., 2011], after constructing the appropriate barrier. We use the same idea as in the proof of Lipschitz continuity; but here we need to prove that $v_{x_0,d_0}(0) \geq C_*$; using the Harnack inequality and the Lipschitz continuity.

Free Boundary Regularity

Theorem (da Silva; R.; Ricarte; Vivas - To appear in Israel J. Math.)
Let u be a viscosity solution to (1) in B_1. Suppose that $0 \in \mathfrak{F}(u)$, $\mathrm{Q}(0) = 1$ and $F(0, X)$ is uniformly elliptic. Then, there exists a universal constant $\tilde{\varepsilon} > 0$ such that, if the graph of u is $\tilde{\varepsilon}$-flat in B_1, i.e.

$$(x_n - \tilde{\varepsilon})^+ \le u(x) \le (x_n + \tilde{\varepsilon})^+ \quad \text{for} \;\; x \in B_1,$$

and

$$\max\left\{\|f\|_{L^\infty(B_1)}, \;\; [\mathrm{Q}]_{C^{0,\alpha}(B_1)}, \|F\|_{C^\omega(B_1)}\right\} \le \tilde{\varepsilon},$$

then $\mathfrak{F}(u)$ is $C^{1,\beta}$ in $B_{1/2}$ for some (universal) $\beta \in (0,1)$.

The proof of the previous theorem is based on an *improvement of flatness* property for the graph of a solution u: if the graph of u oscillates away ε from a hyperplane in B_1 then in B_{δ_0} it oscillates $\frac{\delta_0\varepsilon}{2}$ away from possibly a different hyperplane.

The proof of the previous theorem is based on an *improvement of flatness* property for the graph of a solution u: if the graph of u oscillates away ε from a hyperplane in B_1 then in B_{δ_0} it oscillates $\frac{\delta_0 \varepsilon}{2}$ away from possibly a different hyperplane. The fundamental tools to achieve this property are a Harnack type inequality and characterizing of limiting solutions.

- De Silva - (Interfaces Free Bound. - 2011)

In the proof of the Harnack inequality we have some difficulties to overcome, since the structure of the operator $\mathcal{G}_{p,q}[u] := \mathcal{H}(x, \nabla u)F(x, D^2 u)$ requires some non-trivial adaptations.

In the proof of the Harnack inequality we have some difficulties to overcome, since the structure of the operator $\mathcal{G}_{p,q}[u] := \mathcal{H}(x, \nabla u)F(x, D^2 u)$ requires some non-trivial adaptations. In fact, if ℓ is an affine function and u is a solution to the problem

$$\mathcal{H}(x, \nabla u)F(x, D^2 u) = f(x) \quad \text{in } B_r(x_0), \quad \text{where } x_0 = \frac{e_n}{10}, \tag{2}$$

we can not conclude that $u + \ell$ is a solution to the equation (2) yet. In contrast, for $p = q = 0$ we know $u + \ell$ is still solution for (2). In effect, De Silva have used this fact, thereby allowing us to apply the Harnack inequality for $v(x) = u(x) - x_n$, which plays a crucial role in reaching an *improvement of flatness* for the graph of u.

Lemma (Improvement of flatness)

Let u be a viscosity solution to (1) in Ω under assumptions

$$\|f\|_{L^\infty(\Omega)} \leq \varepsilon^2 \quad \text{and} \quad \|Q-1\|_{L^\infty(\Omega)} \leq \varepsilon^2$$

with $0 \in \mathfrak{F}(u)$ and assume it satisfies

$$(x_n - \varepsilon)^+ \leq u(x) \leq (x_n + \varepsilon)^+ \quad \textit{for } x \in B_1.$$

Then there exists a universal constant $r_0 > 0$ such that if $0 < r \leq r_0$ and $0 < \varepsilon \leq \varepsilon_0$ (with ε_0 depending on r), then

$$\left(\langle x, \nu \rangle - r\frac{\varepsilon}{2}\right)^+ \leq u(x) \leq \left(\langle x, \nu \rangle + r\frac{\varepsilon}{2}\right)^+ \quad x \in B_r,$$

for some $\nu \in \mathbb{S}^{n-1}$ (unity sphere) and $|\nu - e_n| \leq C\varepsilon^2$ for a universal constant $C > 0$.

Sketch of the proof (Flatness implies $C^{1,\beta}$):

- Let u be a viscosity solution to the free boundary problem

$$\begin{cases} \mathcal{H}(x,\nabla u)F(x,D^2u) &= f(x) \quad \text{in} \quad B_1^+(u), \\ |\nabla u| &= \mathrm{Q}(x) \quad \text{on} \quad \mathfrak{F}(u) \end{cases}$$

with $0 \in \mathfrak{F}(u)$ and $\mathrm{Q}(0) = 1$.

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with $0 \in \mathfrak{F}(u)$ and $\mathrm{Q}(0) = 1$.

- Assume further that

$$(x_n - \tilde{\varepsilon})^+ \leq u(x) \leq (x_n + \tilde{\varepsilon})^+ \quad \text{for} \quad x \in B_1,$$

and

$$\max\left\{\|f\|_{L^\infty(B_1)}, \quad [\mathrm{Q}]_{C^{0,\alpha}(B_1)}, \|F\|_{C^\omega(B_1)}\right\} \leq \tilde{\varepsilon},$$

with $\tilde{\varepsilon} > 0$ to be fixed.

- Fix $\bar{r} > 0$ a universal constant such that

$$\bar{r} \leq \min\left\{ r_0,\ \left(\frac{1}{4}\right)^{\frac{1}{\alpha}} \right\},$$

where r_0 comes from the improvement of flatness Lemma. After chosen $\bar{r}$, let $\varepsilon_0 := \varepsilon_0(\bar{r})$ be the constant given by improvement of flatness Lemma.

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$$\tilde{\varepsilon} := \varepsilon_0^2 \quad \text{and} \quad \varepsilon_k := 2^{-k}\varepsilon_0.$$

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$$\tilde{\varepsilon} := \varepsilon_0^2 \quad \text{and} \quad \varepsilon_k := 2^{-k}\varepsilon_0.$$

- The choice of $\tilde{\varepsilon}$ ensures that

$$(x_n - \varepsilon_0)^+ \leq u(x) \leq (x_n + \varepsilon_0)^+ \quad \text{in} \quad B_1.$$

which implies by the improvement of flatness Lemma that there exists ν_1 with $|\nu_1| = 1$ and $|\nu_1 - e_n| \leq \mathrm{C}{\varepsilon_0}^2$ such that

$$\left(\langle x, \nu_1\rangle - \bar{r}\frac{\varepsilon_0}{2}\right)^+ \leq u(x) \leq \left(\langle x, \nu_1\rangle + \bar{r}\frac{\varepsilon_0}{2}\right)^+ \quad \text{in} \quad B_{\bar{r}}.$$

- Now, consider the sequence of re-scaling profiles $u_k : B_1 \to \mathbb{R}$ given by

$$u_k(x) := \frac{u(\lambda_k x)}{\lambda_k}$$

with $\lambda_k = \overline{r}^k$, $k = 0, 1, 2, \ldots$, for a fixed $\overline{r}$ as previously.

- Now, consider the sequence of re-scaling profiles $u_k : B_1 \to \mathbb{R}$ given by

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with $\lambda_k = \bar{r}^k$, $k = 0, 1, 2, \ldots$, for a fixed $\bar{r}$ as previously.

- u_k fulfils in the viscosity sense the following free boundary problem

$$\left\{ \begin{array}{rclcl} \mathcal{H}(\lambda_k x, \nabla u_k) F_k(x, D^2 u_k) & = & f_k(x) & \text{in} & B_1^+(u_k), \\ |\nabla u_k| & = & \mathrm{Q}_k & \text{on} & \mathfrak{F}(u_k), \end{array} \right.$$

where

$$\left\{ \begin{array}{rcl} F_x(x, \mathrm{X}) & := & \lambda_k F\left(\lambda_k x, \lambda_k^{-1} \mathrm{X}\right) \\ \mathcal{H}_k(x, \xi) & := & \mathcal{H}\left(\lambda_k x, \xi\right) \\ \mathfrak{a}_k(x) & := & \mathfrak{a}(\lambda_k x) \\ f_k(x) & := & \lambda_k f(\lambda_k x) \\ \mathrm{Q}_k(x) & := & \mathrm{Q}(\lambda_k x). \end{array} \right.$$

where $F_k, \mathcal{H}_k$ and $\mathfrak{a}_k$ satisfy the structural assumptions.

- We prove that in B_1 we have

$$|f_k(x)| \leq \varepsilon_k^2 \quad \text{and} \quad |Q_k(x) - 1| \leq \varepsilon_k^2.$$

- We prove that in B_1 we have

$$|f_k(x)| \leq \varepsilon_k^2 \quad \text{and} \quad |Q_k(x) - 1| \leq \varepsilon_k^2.$$

- We iterate the above argument and obtain that

$$(\langle x, \nu_k \rangle - \varepsilon_k)^+ \leq u_k(x) \leq (\langle x, \nu_k \rangle + \varepsilon_k)^+ \quad \text{in} \quad B_1,$$

with $|\nu_k| = 1$, $|\nu_k - \nu_{k+1}| \leq C\varepsilon_k$ (with $\nu_0 = e_n$).

- Therefore

$$\left(\langle x, \nu_k\rangle - \frac{\varepsilon_0}{2^k}\bar{r}^k\right)^+ \leq u(x) \leq \left(\langle x, \nu_k\rangle + \frac{\varepsilon_0}{2^k}\bar{r}^k\right)^+ \quad \text{in} \quad B_{\bar{r}^k} \qquad (3)$$

  with

$$|\nu_{k+1} - \nu_k| \leq \mathrm{C}\frac{\varepsilon_0}{2^k}.$$

- Therefore

$$\left(\langle x,\nu_k\rangle - \frac{\varepsilon_0}{2^k}\bar{r}^k\right)^+ \leq u(x) \leq \left(\langle x,\nu_k\rangle + \frac{\varepsilon_0}{2^k}\bar{r}^k\right)^+ \quad \text{in} \quad B_{\bar{r}^k} \quad (3)$$

with

$$|\nu_{k+1} - \nu_k| \leq \mathrm{C}\frac{\varepsilon_0}{2^k}.$$

- Furthermore, (3) implies that

$$\partial\{u>0\} \cap B_{\bar{r}^k} \subset \left\{|\langle x,\nu_k\rangle| \leq \frac{\varepsilon_0}{2^k}\bar{r}^k\right\},$$

which implies that $B_{3/4} \cap \mathfrak{F}(u)$ is a $C^{1,\beta}$ graph.

Lipschitz implies $C^{1,\beta}$

Theorem (da Silva; R.; Ricarte; Vivas - To appear in Israel J. Math.)
Let u be a viscosity solution for the free boundary problem (1). Assume further that $0 \in \mathfrak{F}(u)$, $f \in L^{\infty}(B_1)$ is continuous in $B_1^+(u)$ and $\mathrm{Q}(0) > 0$. If $\mathfrak{F}(u)$ is a Lipschitz graph in a neighborhood of 0, then $\mathfrak{F}(u)$ is $C^{1,\beta}$ in a (smaller) neighborhood of 0.

Comments about the proof:

We use a blow-up argument from the previous theorem and the approach used in [Caffarelli - Rev. Mat. Iberoamericana, 1987]:

Comments about the proof:

We use a blow-up argument from the previous theorem and the approach used in [Caffarelli - Rev. Mat. Iberoamericana, 1987]:

- We consider the re-scaled functions

$$u_k(x) := u_{\delta_k}(x) = \frac{u(\delta_k x)}{\delta_k},$$

  with $\delta_k \to 0$ as $k \to \infty$;

- Because of the non-degeneracy and the Lipschitz continuity for u_k's, $u_k \to u_\infty$ uniformly on compact sets. Then, we can prove that u_∞ solves

$$\begin{cases} \mathrm{F}_\infty(D^2 u_\infty) & = & 0 & \text{in} & \{u_\infty > 0\}, \\ |\nabla u_\infty| & = & 1 & \text{on} & \mathfrak{F}(u_\infty), \end{cases}$$

with $\mathfrak{F}(u_\infty)$ Lipschitz continuous (via compactness lemma and cutting lemma).

- Also, we can prove that u_∞ is a one-phase solution, i.e. up to rotations,
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- Thus, for k large enough we have

$$\|u_k - u_\infty\|_{L^\infty(B_1)} \leq \tilde{\varepsilon},$$

where $\tilde{\varepsilon}$ comes from the previous lemma;
- Therefore, u_k fulfils $\tilde{\varepsilon}$-flat in B_1, which implies that $\mathfrak{F}(u_k)$ is a graph $C^{1,\beta}$ and consequently $\mathfrak{F}(u)$ are $C^{1,\beta}$, for some $\beta \in (0,1)$.

Some extensions

We can extend our results for nonlinear elliptic equations with non-homogeneous term as follows.

We can extend our results for nonlinear elliptic equations with non-homogeneous term as follows.

- **Multi degenerate operators in non-divergence form.**
  An extension of our results holds to general multi-degenerate fully nonlinear models given by

$$\mathcal{G}(x, Du, D^2u) := \left(|Du|^p + \sum_{i=1}^{N} \mathfrak{a}_i(x)|Du|^{q_i}\right) F(x, D^2u),$$

  where $0 \leq \mathfrak{a}_i \in C^0(\Omega)$, $i \in \{1, \cdots, N\}$, and $0 < p \leq q_1 \leq \cdots \leq q_N < \infty$, which are a natural non-variational counterpart of certain multi-phase variational problems treated in [De Filippis, Oh - J. Differential Equations, 2019].

- **Doubly degenerate $(p,q)-$Laplacian in non-divergence form.**

  Other interesting class of degenerate operators where our results work out is the double degenerate $p-$Laplacian type operators, in non-divergence form, for $2 < p_0 \le q_0 < \infty$ and $1 < p < \infty$:

$$\mathcal{G}_{p_0,q_0}(x,\xi,X) = \mathcal{H}_{p_0,q_0}(x,\xi)F_p(\xi,X)$$

  where

$$\mathcal{H}_{p_0,q_0}(x,\xi) := |\xi|^{p_0-2} + \mathfrak{a}(x)|\xi|^{q_0-2}$$

  and

$$F_p(\xi,X) := \mathrm{Tr}\left[\left(\mathrm{Id}_n + (p-2)\frac{\xi\otimes\xi}{|\xi|^2}\right)X\right]$$

  is the well-known Normalized $p-$Laplacian operator. See [Attouchi, Parviainen, Ruosteenoja - J. Math. Pures Appl., 2017] and [da Silva, Ricarte - Calc. Var. PDE, 2020].

- **Fully nonlinear models with non-standard growth.** Other example we have in mind is the class of variable-exponent, degenerate elliptic equations in non-divergence form, which is, in some extent, the non-variational counterpart of certain non-homogeneous functionals satisfying nonstandard growth conditions:

$$\mathcal{G}_{p(x),q(x)}(x, \xi, X) := \left(|\xi|^{p(x)} + \mathfrak{a}(x)|\xi|^{q(x)}\right) F(x, \mathrm{X}),$$

for rather general exponents $p, q \in C^0(\Omega; (0, \infty)$. See [Bronzi, Pimentel, Rampasso, Teixeira - J. Funct. Anal., 2020].

OBRIGADA PELA ATENÇÃO! =)