"Seminários do grupo de Análise"

Universidade Estadual de Campinas

On critical problems with nonstandard growth

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References

In the seminal paper

- V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. (1986)

Zhikov introduced the double phase integral functional

$$u \mapsto \int_\Omega (|\nabla u|^p + a(x)|\nabla u|^q)dx, \quad 1 < p < q, \quad 0 \leq a(\cdot) \leq L, \tag{1}$$

in order to provide models of strongly anisotropic materials in the context of homogenization and elasticity. This functional revealed to be important also in the study of duality theory and of the Lavrentiev phenomenon.

Furthermore, functional (1) belongs to the class of the integral functionals with nonstandard growth condition, according to Marcellini's terminology. That is, the integrand $\mathcal{H}(x,\xi) = |\xi|^p + a(x)|\xi|^q$ has an unbalanced growth, as

$$|\xi|^p \leq \mathcal{H}(x,\xi) \leq b(1+|\xi|^q) \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N,$$

with $b > 0$.

References

- P. Marcellini, *On the definition and the lower semicontinuity of certain quasiconvex integrals*, Ann. Inst. H. Poincaré Anal. Non Linéaire (1986)
- P. Marcellini, *Regularity and existence of solutions of elliptic equations with (p, q)-growth conditions*, J. Differential Equations (1991)
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- V.V. Zhikov, *On some variational problems*, Russian J. Math. Phys. (1997)
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Part 1:

A critical (p, q) system

(p, q) SYSTEM

- G.M. Figueiredo, *Existence of positive solutions for a class of p&q elliptic problems with critical growth on* $\mathbb{R}^N$, J. Math. Anal. Appl. (2011)

In the above paper, they studied the equation in $\mathbb{R}^N$

$$-\mathrm{div}\big(A(|\nabla u|)\nabla u\big) + B(|u|)u = \lambda f(u) + |u|^{\wp^*-2}u,$$

where

- A and B have a nonstandard (p, q) growth, with $p < q$;
- λ is a real parameter;
- f is a subcritical continuous function;
- $\wp^* = N\wp/(N - \wp)$, with $1 < \wp < N$, is the maximum critical Sobolev exponent, according to the behaviour of A.

(p,q) SYSTEM

- A. Fiscella, P. Pucci, *(p,q) systems with critical terms in $\mathbb{R}^N$*, Nonlinear Anal. (2018)

We generalize the result, studying the following system in $\mathbb{R}^N$

$$\begin{cases} -\mathrm{div}\big(A(|\nabla u|)\nabla u\big) + B(|u|)u - \sigma \dfrac{|u|^{\wp-2}u}{|x|^{\wp}} = \lambda H_u(x,u,v) \\ \qquad\qquad\qquad + \dfrac{\alpha}{\wp^*}|v|^{\beta}|u|^{\alpha-2}u, \\ -\mathrm{div}\big(A(|\nabla v|)\nabla v\big) + B(|v|)v - \sigma \dfrac{|v|^{\wp-2}v}{|x|^{\wp}} = \lambda H_v(x,u,v) \\ \qquad\qquad\qquad + \dfrac{\beta}{\wp^*}|u|^{\alpha}|v|^{\beta-2}v, \end{cases} \qquad (\mathcal{S})$$

where

- $\alpha > 1$ and $\beta > 1$, with $\alpha + \beta = \wp^*$ and $1 < \wp < N$;
- σ and λ are real parameters.

Concerning the elliptic part, we assume

(C_1) *A and B are strictly positive and strictly increasing functions of class $C^1(\mathbb{R}^+)$.*

Clearly, (C_1) implies that $tA(t) \to 0$ and $tB(t) \to 0$ as $t \to 0^+$.

Let us introduce for simplicity the functions $\mathcal{A}$ and $\mathcal{B}$ as the potentials, which are 0 at 0 and which are obtained by integration from

$$\mathcal{A}'(t) = tA(t), \quad \mathcal{B}'(t) = tB(t) \quad \text{for all } t \in \mathbb{R}_0^+,$$

where $tA(t)$ and $tB(t)$ are defined to be 0 at 0 thanks to (C_1).

(C_2) *there exist constants* a_0, $\mathfrak{a}_0$, b_0, $\mathfrak{b}_0$ *strictly positive, with* $a_0 \leq 1$, a_1, $\mathfrak{a}_1$, b_1, $\mathfrak{b}_1$ *nonnegative, with the property that* $\mathfrak{a}_1 > 0$ *implies* $\mathfrak{b}_1 > 0$, $a_1 > 0$ *and* $b_1 > 0$, *and there are exponents* p *and* q, *with* $1 < p < q < \wp^*$, *where* $1 < \wp < N$ *with*

$$\wp = \begin{cases} p, & \text{if } \mathfrak{a}_1 = 0, \\ q, & \text{if } \mathfrak{a}_1 > 0, \end{cases}$$

such that for all $t \in \mathbb{R}_0^+$

$$a_0 t^{p-1} + \mathbb{1}_{\mathbb{R}^+}(\mathfrak{a}_1) a_1 t^{q-1} \leq \mathcal{A}'(t) \leq \mathfrak{a}_0 t^{p-1} + \mathfrak{a}_1 t^{q-1},$$
$$b_0 t^{p-1} + \mathbb{1}_{\mathbb{R}^+}(\mathfrak{b}_1) b_1 t^{q-1} \leq \mathcal{B}'(t) \leq \mathfrak{b}_0 t^{p-1} + \mathfrak{b}_1 t^{q-1},$$

where $\mathbb{1}_E$ *is the characteristic function of a Lebesgue measurable subset* E *of* $\mathbb{R}$.

(C_3) *there exist constants* θ *and* ϑ, *with* $\wp \leq \min\{\theta, \vartheta\} < \wp^*$, *such that*

$$\theta \mathcal{A}(t) \geq t\mathcal{A}'(t), \qquad \vartheta \mathcal{B}(t) \geq t\mathcal{B}'(t) \quad \text{for all } t \in \mathbb{R}_0^+$$

holds.

Examples

We just present few examples which illustrate the general systems covered in this paper under the assumptions (C_1)-(C_3). In the examples we tacitly suppose that $1 < p < q < \wp^*$ and $1 < \wp < N$, without mentioning.

If $\mathcal{A}(t) = \mathcal{B}(t) = t^p/p,\ t \in \mathbb{R}_0^+,$ then $a_0 = \mathfrak{a}_0 = b_0 = \mathfrak{b}_0 = 1,$ $a_1 = \mathfrak{a}_1 = b_1 = \mathfrak{b}_1 = 0,\ \wp = p,\ \theta = \vartheta = p,\ \alpha + \beta = p^*$ and $(\mathcal{S})$ reduces to

$$\begin{cases} -\Delta_p u + |u|^{p-2}u - \sigma \dfrac{|u|^{p-2}u}{|x|^p} = \lambda H_u(x,u,v) + \dfrac{\alpha}{p^*}|v|^{\beta}|u|^{\alpha-2}u, \\ -\Delta_p v + |v|^{p-2}v - \sigma \dfrac{|v|^{p-2}v}{|x|^p} = \lambda H_v(x,u,v) + \dfrac{\beta}{p^*}|u|^{\alpha}|v|^{\beta-2}v, \end{cases}$$

where $\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$.

Examples

Similarly, for $\mathcal{A}(t) = \mathcal{B}(t) = t^p/p + t^q/q,\ t \in \mathbb{R}_0^+,$ then $a_0 = \mathfrak{a}_0 = a_1 = \mathfrak{a}_1 = b_0 = \mathfrak{b}_0 = b_1 = \mathfrak{b}_1 = 1,\ \wp = q,\ \theta = \vartheta = q,$ $\alpha + \beta = q^*$ and $(\mathcal{S})$ becomes

$$\begin{cases} -\Delta_p u - \Delta_q u + |u|^{p-2}u + |u|^{q-2}u - \sigma \dfrac{|u|^{q-2}u}{|x|^q} = \lambda H_u(x,u,v) \\ \qquad\qquad + \dfrac{\alpha}{q^*}|v|^\beta |u|^{\alpha-2}u, \\ -\Delta_p v - \Delta_q v + |v|^{p-2}v + |v|^{q-2}v - \sigma \dfrac{|v|^{q-2}v}{|x|^q} = \lambda H_v(x,u,v) \\ \qquad\qquad + \dfrac{\beta}{q^*}|u|^\alpha |v|^{\beta-2}v. \end{cases}$$

Examples

For $\mathcal{A}(t) = t^p/p + \frac{1}{2}(1+t^p)^{2/p} - 1/2$ and $\mathcal{B}(t) = t^p/p + c\,t^q/q$, $t \in \mathbb{R}_0^+$, $c \geq 0$, and whenever $2 \leq p < q$, then $a_0 = b_0 = \mathfrak{b}_0 = 1$, $\mathfrak{a}_0 = 2$, $a_1 = \mathfrak{a}_1 = 0$, $\mathfrak{b}_1 = c$, $\wp = p$, $\theta = p$ and $\vartheta = p$ if $c = 0$, while $\vartheta = q$ if $c > 0$, $\alpha + \beta = p^*$ and $(\mathcal{S})$ reads as

$$\begin{cases} -\Delta_p u - \operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{\left(1+|\nabla u|^p\right)^{1-2/p}}\right) + |u|^{p-2}u + c\,|u|^{q-2}u \\ \qquad - \sigma \dfrac{|u|^{p-2}u}{|x|^p} = \lambda H_u(x,u,v) + \dfrac{\alpha}{p^*}|v|^{\beta}|u|^{\alpha-2}u, \\ -\Delta_p v - \operatorname{div}\left(\dfrac{|\nabla v|^{p-2}\nabla v}{\left(1+|\nabla v|^p\right)^{1-2/p}}\right) + |v|^{p-2}v + c\,|v|^{q-2}v \\ \qquad - \sigma \dfrac{|v|^{p-2}v}{|x|^p} = \lambda H_v(x,u,v) + \dfrac{\beta}{p^*}|u|^{\alpha}|v|^{\beta-2}v. \end{cases}$$

EXAMPLES

If $\mathcal{A}(t) = \sqrt{1+t^2} - 1 + t^4/4$ and $\mathcal{B}(t) = t^2/2 + t^4/4$, $t \in \mathbb{R}_0^+$, with $2 = p < q = 4$, then $a_0 = b_0 = b_1 = \mathfrak{a}_0 = \mathfrak{a}_1 = \mathfrak{b}_0 = \mathfrak{b}_1 = 1$, $a_1 = 1/2$, $\wp = q = 4$, $\theta = \vartheta = 4$, $\alpha + \beta = q^* = 4^*$ and $(\mathcal{S})$ reads as

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - \Delta_4 u + u + u^3 - \sigma \dfrac{u^3}{|x|^4} = \lambda H_u(x,u,v) \\ \qquad\qquad\qquad\qquad + \dfrac{\alpha}{4^*}|v|^\beta |u|^{\alpha-2} u, \\ -\operatorname{div}\left(\dfrac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) - \Delta_4 v + v + v^3 - \sigma \dfrac{v^3}{|x|^4} = \lambda H_v(x,u,v) \\ \qquad\qquad\qquad\qquad + \dfrac{\beta}{4^*}|u|^\alpha |v|^{\beta-2} v. \end{cases}$$

Examples

As in the example above, taking $\mathcal{A}(t) = t \arctan t - \log \sqrt{1+t^2} + t^4/4$ and $\mathcal{B}(t) = t^2/2 + t^4/4$, $t \in \mathbb{R}_0^+$, with $2 = p < q = 4$, then $a_0 = b_0 = b_1 = \mathfrak{a}_0 = \mathfrak{a}_1 = \mathfrak{b}_0 = \mathfrak{b}_1 = 1$, $a_1 = 2/3$, $\wp = q = 4$, $\theta = \vartheta = 4$, $\alpha + \beta = q^* = 4^*$ and $(\mathcal{S})$ reads as

$$
\begin{cases}
-\operatorname{div}\left(\dfrac{\arctan|\nabla u|}{|\nabla u|}\nabla u\right) - \Delta_4 u + u + u^3 - \sigma\dfrac{u^3}{|x|^4} = \lambda H_u(x,u,v) \\
\qquad\qquad\qquad\qquad\qquad + \dfrac{\alpha}{4^*}|v|^\beta |u|^{\alpha-2}u, \\
-\operatorname{div}\left(\dfrac{\arctan|\nabla v|}{|\nabla v|}\nabla v\right) - \Delta_4 v + v + v^3 - \sigma\dfrac{v^3}{|x|^4} = \lambda H_v(x,u,v) \\
\qquad\qquad\qquad\qquad\qquad + \dfrac{\beta}{4^*}|u|^\alpha |v|^{\beta-2}v.
\end{cases}
$$

While the nonlinearity H in $(\mathcal{S})$ is a *Carathéodory* function satisfying

(H) *For a.e.* $x \in \mathbb{R}^N$ *it results* $H(x,\cdot,\cdot) \in C^1(\mathbb{R}^2)$, $H(x,\cdot,\cdot) \geq 0$ *in* $\mathbb{R}^2$, $H_u(x,u,v) = 0$ *for all* $u \leq 0$ *and* $v \in \mathbb{R}$, *while* $H_v(x,u,v) = 0$ *for all* $u \in \mathbb{R}$ *and* $v \leq 0$.

Furthermore, there exist $\mathfrak{m}$, m, γ *such that* $\wp < \mathfrak{m} < m < \wp^*$, $\max\{\theta, \vartheta\} < \gamma < \wp^*$ *and for every* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *for which the inequality*

$$|H_z(x,z)| \leq \mathfrak{m}\varepsilon|z|^{\mathfrak{m}-1} + mC_\varepsilon|z|^{m-1} \quad \textit{for any } z \in \mathbb{R}^2,$$

where $z = (u,v)$, $|z| = \sqrt{u^2+v^2}$, $H_z = (H_u, H_v)$, *and also the inequalities*

$$0 \leq \gamma H(x,z) \leq H_z(x,z) \cdot z \quad \textit{for any } z \in \mathbb{R}^2,$$

hold for a.e. $x \in \mathbb{R}^N$, *where* θ, ϑ *are given in* (C_3).

Finally, $H(x,u,v) > 0$ *for a.e.* $x \in \mathbb{R}^N$ *and all* $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Functional space

The best Hardy constant, called $\mathcal{H}_\wp = \mathcal{H}(\wp, N)$, is given by

$$\mathcal{H}_\wp := \inf_{\substack{u \in D^{1,\wp}(\mathbb{R}^N) \\ u \neq 0}} \frac{\|\nabla u\|_\wp^\wp}{\|u\|_{H_\wp}^\wp}, \qquad \text{where} \qquad \|u\|_{H_\wp} := \left(\int_{\mathbb{R}^N} |u|^\wp \frac{dx}{|x|^\wp} \right)^{1/\wp},$$

and $D^{1,\wp}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$, with respect to the norm $\|\nabla u\|_\wp = \left(\int_{\mathbb{R}^N} |\nabla u|^\wp dx \right)^{1/\wp}$.

Since $\wp = p$ if $\mathfrak{a}_1 = 0$, while $\wp = q$ if $\mathfrak{a}_1 > 0$, the natural space where finding solutions of $(\mathcal{S})$ is

$$W := \left(W^{1,p}(\mathbb{R}^N) \cap W^{1,\wp}(\mathbb{R}^N) \right) \times \left(W^{1,p}(\mathbb{R}^N) \cap W^{1,\wp}(\mathbb{R}^N) \right),$$

endowed with the norm

$$\|(u,v)\| := \|u\|_{W^{1,p}} + \|v\|_{W^{1,p}} + \mathbb{1}_{\mathbb{R}^+}(\mathfrak{a}_1) \left(\|u\|_{W^{1,q}} + \|v\|_{W^{1,q}} \right),$$

where $\|u\|_{W^{1,\mathfrak{p}}} = \|u\|_\mathfrak{p} + \|\nabla u\|_\mathfrak{p}$ for all $u \in W^{1,\mathfrak{p}}(\mathbb{R}^N)$ and any $\mathfrak{p} > 1$.

Main result

Theorem 1 (Fiscella & Pucci, NA 2018)

Suppose that (C_1)-(C_3) and (H) hold.

Then, for any $\sigma \in (-\infty, \mathfrak{c}_a \mathcal{H}_\wp)$ there exists $\widetilde{\lambda} = \widetilde{\lambda}(\sigma) > 0$ such that system

$$\begin{cases} -\operatorname{div}\big(A(|\nabla u|)\nabla u\big) + B(|u|)u - \sigma \dfrac{|u|^{\wp-2}u}{|x|^\wp} = \lambda H_u(x,u,v) \\ \qquad\qquad + \dfrac{\alpha}{\wp^*}|v|^\beta |u|^{\alpha-2}u, \\ -\operatorname{div}\big(A(|\nabla v|)\nabla v\big) + B(|v|)v - \sigma \dfrac{|v|^{\wp-2}v}{|x|^\wp} = \lambda H_v(x,u,v) \\ \qquad\qquad + \dfrac{\beta}{\wp^*}|u|^\alpha |v|^{\beta-2}v, \end{cases} \qquad (\mathcal{S})$$

admits at least one nontrivial solution $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ in W for all $\lambda \geq \widetilde{\lambda}$. Moreover, each component of $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ is nontrivial and

$$\lim_{\lambda\to\infty} \|(u_{\sigma,\lambda}, v_{\sigma,\lambda})\| = 0. \qquad (2)$$

Sketch of the proof

Clearly, the weak solutions of $(\mathcal{S})$ are exactly the critical points of the Euler-Lagrange functional $\mathcal{I}_{\sigma,\lambda} : W \to \mathbb{R}$, given by

$$\begin{aligned}\mathcal{I}_{\sigma,\lambda}(u) := &\int_{\mathbb{R}^N} \big[\mathcal{A}(|\nabla u|) + \mathcal{A}(|\nabla v|)\big] dx + \int_{\mathbb{R}^N} \big[\mathcal{B}(|u|) + \mathcal{B}(|v|)\big] dx \\ &- \frac{\sigma}{\wp} \int_{\mathbb{R}^N} \left(|u|^{\wp} + |v|^{\wp}\right) \frac{dx}{|x|^{\wp}} - \lambda \int_{\mathbb{R}^N} H(x,u,v) dx \\ &- \frac{1}{\wp^*} \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta} dx,\end{aligned}$$

which is well defined and of class C^1 on W.

We can set the mountain pass level

$$c_{\sigma,\lambda} = \inf_{\xi\in\Gamma} \max_{t\in[0,1]} \mathcal{I}_{\sigma,\lambda}(\xi(t)),$$

where

$$\Gamma = \left\{\xi \in C([0,1], W) \ : \ \xi(0) = (0,0), \ \mathcal{I}_{\sigma,\lambda}(\xi(1)) < 0\right\}.$$

Sketch of the proof

In order to prove the delicate Palais-Smale condition at $c_{\sigma,\lambda}$, for a Palais-Smale sequence $\{(u_n, v_n)\}_n \subset W$ we have to show that:

(i) up to a subsequence, $(u_n, v_n) \rightharpoonup (u_{\sigma,\lambda}, v_{\sigma,\lambda})$ in W as $n \to \infty$;

(ii) by concentration-compactness principle, there exists $\lambda^* = \lambda^*(\sigma) > 0$ such that the weak limit $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ is a solution of $(\mathcal{S})$ for all $\lambda \geq \lambda^*$;

(iii) since $c_{\sigma,\lambda} \to 0$ as $\lambda \to \infty$, the set $\{(u_{\sigma,\lambda}, v_{\sigma,\lambda})\}_{\lambda \geq \lambda^*}$ satisfies the asymptotic property (2);

(iv) there exists $\widetilde{\lambda} = \widetilde{\lambda}(\sigma) \geq \lambda^*$ such that $(u_n, v_n) \to (u_{\sigma,\lambda}, v_{\sigma,\lambda})$ in W as $n \to \infty$ for all $\lambda \geq \widetilde{\lambda}$.

Theorem 2 (Fiscella & Pucci, NA 2018)

Suppose that A satisfies (C_1), B verifies condition

(B) *$B \in C(\mathbb{R}^+)$ and $t \mapsto tB(t)$ is strictly increasing in $\mathbb{R}^+$, with $tB(t) \to 0$ as $t \to 0^+$.*

and that (C_2)-(C_3) and (H) hold.

Then, there exists $\lambda^* > 0$ such that system

$$\begin{cases} -\mathrm{div}\big(A(|\nabla u|)\nabla u\big) + B(|u|)u = \lambda H_u(u,v) + \dfrac{\alpha}{\wp^*}|v|^\beta |u|^{\alpha-2}u, \\ -\mathrm{div}\big(A(|\nabla v|)\nabla v\big) + B(|v|)v = \lambda H_v(u,v) + \dfrac{\beta}{\wp^*}|u|^\alpha |v|^{\beta-2}v, \end{cases} \qquad (\mathcal{S}')$$

admits at least one nontrivial solution (u_λ, v_λ) in W for all $\lambda \geq \lambda^*$. Moreover, each component of (u_λ, v_λ) is nontrivial and

$$\lim_{\lambda\to\infty} \|(u_\lambda, v_\lambda)\| = 0$$

Sketch of the proof

Proposition

For any $\sigma \in (-\infty, \mathfrak{c}_a \mathcal{H}_\wp)$ and $\lambda > 0$ let $\{(u_n, v_n)\}_n \subset W$ be a Palais-Smale sequence at level c_λ such that $(u_n, v_n) \rightharpoonup (0,0)$ in W as $n \to \infty$. Then

(i) either $(u_n, v_n) \to (0,0)$ in W,

(ii) or there exists $R > 0$ and a sequence $(y_n)_n \in \mathbb{R}^N$ such that

$$\limsup_{n\to\infty} \int_{B_R(y_n)} \left(|u_n|^p + |v_n|^p\right) dx > 0.$$

Moreover, $(y_n)_n$ is not bounded in $\mathbb{R}^N$.

By this proposition, we use a compactness argument for the translated sequence $\{(\widetilde{u}_n, \widetilde{v}_n)\}_n$, with $\widetilde{u}_n = u_n(\cdot + y_n)$, $\widetilde{v}_n = v_n(\cdot + y_n)$. This fact forces the main functions in $(\mathcal{S}')$ are independent of x.

Part 2:

A Hardy double phase problem

HARDY PROBLEM

- A. Fiscella, *A double phase problem involving Hardy potentials*, Appl. Math. Optim. (2022)

We study the following problem

$$\begin{cases} -\operatorname{div}(A(x,\nabla u)) - \sigma \left(\dfrac{|u|^{p-2}u}{|x|^p} + a(x)\dfrac{|u|^{q-2}u}{|x|^q} \right) = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \qquad (\mathscr{P}_H)$$

where the main operator is the so-called double phase operator, set as

$$\operatorname{div}(A(x,\nabla u)) = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right).$$

Here, we assume that $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary and $0 \in \Omega$, σ is a real parameter, $1 < p < q < N$ and

(a_1) $\dfrac{q}{p} < 1 + \dfrac{1}{N}$, *while* $a : \overline{\Omega} \to [0,\infty)$ *is Lipschitz continuous.*

Here $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a *Carathéodory* function verifying

(g_1) *there exists an exponent* $r \in (q, p^*)$, *with* $p^* = Np/(N-p)$, *such that for any* $\varepsilon > 0$ *there exists* $c_\varepsilon = c(\varepsilon) > 0$ *and*

$$|g(x,t)| \leq q\varepsilon\, |t|^{q-1} + r\delta_\varepsilon\, |t|^{r-1}$$

holds for a.e. $x \in \Omega$ *and any* $t \in \mathbb{R}$;

(g_2) *there exist* $\theta \in (q, p^*)$, $c > 0$ *and* $t_0 \geq 0$ *such that*

$$c \leq \theta G(x,t) \leq t g(x,t)$$

for a.e. $x \in \Omega$ *and any* $|t| \geq t_0$, *where* $G(x,t) = \int_0^t g(x,\tau)d\tau$.

The existence of r and θ are guaranteed by (a_1), which joint with $q > 1$ yields that $q < p^*$.

Functional space

Thanks to (a_1), the function $\mathcal{H} : \Omega \times [0, \infty) \to [0, \infty)$ defined as

$$\mathcal{H}(x,t) := t^p + a(x)t^q, \quad \text{for a.e. } x \in \Omega \text{ and for any } t \in [0,\infty),$$

is a generalized N-function. Therefore, by

- J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Math. 1983

we can define the Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ as

$$L^{\mathcal{H}}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable} \,\middle|\, \varrho_{\mathcal{H}}(u) < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}} := \inf \left\{ \lambda > 0 : \ \varrho_{\mathcal{H}} \left(\frac{u}{\lambda} \right) \le 1 \right\},$$

where $\varrho_{\mathcal{H}}$ denotes the $\mathcal{H}$-modular function, set as

$$\varrho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x, |u|) dx = \int_\Omega \left(|u|^p + a(x)|u|^q \right) dx.$$

Functional space

We point out that

$$\min\{\|u\|_{\mathcal{H}}^{p}, \|u\|_{\mathcal{H}}^{q}\} \leq \varrho_{\mathcal{H}}(u) \leq \max\{\|u\|_{\mathcal{H}}^{p}, \|u\|_{\mathcal{H}}^{q}\}.$$

The related Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega) := \left\{u \in L^{\mathcal{H}}(\Omega) : \ |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{1,\mathcal{H}} := \|u\|_{\mathcal{H}} + \||\nabla u|\|_{\mathcal{H}}.$$

The natural functional space where finding solutions of $(\mathscr{P}_H)$ is $W_0^{1,\mathcal{H}}(\Omega)$, given by the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$ which can be endowed with the norm

$$\|u\| := \||\nabla u|\|_{\mathcal{H}},$$

equivalent to the norm $\|\cdot\|_{1,\mathcal{H}}$, whenever (a_1) holds true, as proved in

- F. Colasuonno, M. Squassina, *Eigenvalues for double phase variational integrals*, Ann. Mat. Pura Appl. (2016)

Functional space

Denoting by the weighted space

$$L^q_a(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable} \;\middle|\; \int_\Omega a(x)|u|^q dx < \infty \right\},$$

endowed with the seminorm

$$\|u\|_{q,a} := \left(\int_\Omega a(x)|u|^q dx \right)^{1/q},$$

we have the following embeddings.

Proposition (Colasuonno & Squassina, AMPA 2016)

(i) $W^{1,\mathcal{H}}_0(\Omega) \hookrightarrow W^{1,r}_0(\Omega)$ for all $r \in [1, p]$;

(ii) $W^{1,\mathcal{H}}_0(\Omega) \hookrightarrow L^r(\Omega)$ for all $r \in [1, p^*]$;

(iii) $W^{1,\mathcal{H}}_0(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ for all $r \in [1, p^*)$;

(iv) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^q_a(\Omega)$;

(v) $L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$.

Hardy problem

In order to handle the Hardy potentials in

$$\begin{cases} -\operatorname{div}(A(x,\nabla u)) - \sigma\left(\dfrac{|u|^{p-2}u}{|x|^p} + a(x)\dfrac{|u|^{q-2}u}{|x|^q}\right) = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{$\mathscr{P}_H$}$$

we need a further assumption

(a_2) *$a(\lambda x) \le a(x)$ for any $\lambda \in (0,1]$ and any $x \in \overline{\Omega}$,*

which guarantees for any $u \in W_0^{1,\mathcal{H}}(\Omega)$ that

$$H_p \|u\|_{H_p}^p \le \|\nabla u\|_p^p, \qquad \text{with } \|u\|_{H_p} := \int_\Omega \frac{|u|^p}{|x|^p} dx$$

$$H_q \|u\|_{H_{q,a}}^q \le \|\nabla u\|_{q,a}^q, \qquad \text{with } \|u\|_{H_{q,a}} := \int_\Omega a(x)\frac{|u|^q}{|x|^q} dx,$$

where constant $H_m := \left(\dfrac{m}{N-m}\right)^{-m}$ for $m = p, q$.

Theorem (Fiscella, AMOP 2022)

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary and $0 \in \Omega$. Let $1 < p < q < N$ and $a(\cdot)$ satisfy (a_1)-(a_2). Let (g_1)-(g_2) hold true.

Then, for any $\sigma \in (-\infty, \min\{H_p, H_q\})$ problem

$$\begin{cases} -\operatorname{div}(A(x,\nabla u)) - \sigma\left(\dfrac{|u|^{p-2}u}{|x|^p} + a(x)\dfrac{|u|^{q-2}u}{|x|^q}\right) = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases} \qquad (\mathscr{P}_H)$$

admits a non-trivial weak solution.

Hardy problem

Theorem (Fiscella, AMOP 2022)

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary and $0 \in \Omega$. Let $1 < p < q < N$ and $a(\cdot)$ satisfy (a_1)-(a_2). Let (g_1)-(g_2) hold true.

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admits a non-trivial weak solution.

The theorem generalizes the existence result stated in

- W. Liu, G. Dai, *Existence and multiplicity results for double phase problem*, J. Differential Equations (2018)

where they considered $(\mathscr{P}_H)$ without Hardy potentials, namely when $\sigma = 0$.

Hardy problem

Indeed, let us consider operator $L_\sigma : W_0^{1,\mathcal{H}}(\Omega) \to \left(W_0^{1,\mathcal{H}}(\Omega)\right)^*$, such that

$$
\begin{aligned}
\langle L_\sigma(u), v\rangle := & \int_\Omega \left(|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}\right) \nabla u \cdot \nabla v \, dx \\
& - \sigma \int_\Omega \left(\frac{|u|^{p-2}u}{|x|^p} v + a(x) \frac{|u|^{q-2}u}{|x|^q} v \right) dx, \quad u, v \in W_0^{1,\mathcal{H}}(\Omega).
\end{aligned}
$$

Remark

Let $2 \leq p < q < N$ and $\sigma \in (-\infty, K_{p,q} \min\{H_p, H_q\})$, with $K_{p,q} \in (0,1)$.

Then, operator L_σ is a mapping of (S) type, that is if $u_n \rightharpoonup u$ in $W_0^{1,\mathcal{H}}(\Omega)$ and

$$\lim_{n\to\infty} \langle L_\sigma(u_n) - L_\sigma(u), u_n - u\rangle = 0$$

then $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$.

Hardy problem

Clearly, the weak solutions of

$$\begin{cases} -\operatorname{div}(A(x,\nabla u)) - \sigma\left(\dfrac{|u|^{p-2}u}{|x|^p} + a(x)\dfrac{|u|^{q-2}u}{|x|^q}\right) = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \qquad (\mathscr{P}_H)$$

are exactly the critical points of the Euler-Lagrange functional $\mathcal{J}_\sigma : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$, given by

$$\mathcal{J}_\sigma(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_{q,a}^q - \sigma\left(\frac{1}{p}\|u\|_{H_p}^p + \frac{1}{q}\|u\|_{H_{q,a}}^q\right) - \int_\Omega G(x,u)dx,$$

which is well defined and of class C^1 on $W_0^{1,\mathcal{H}}(\Omega)$. However, here we have to deal with the non-compactness of the embeddings

$$W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^p(\Omega, |x|^{-p}), \qquad W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^q(\Omega, a(x)|x|^{-q}).$$

Part 3:

Critical Sobolev double phase problems

First critical problem

- C. Farkas, P. Winkert, *An existence result for singular Finsler double phase problems*, J. Differential Equations (2021)

They considered the following problem

$$\begin{cases} -\operatorname{div}(A_F(x,\nabla u)) = u^{p^*-1} + \lambda\left(u^{\gamma-1} + g(x,u)\right) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

driven by the Finsler double phase operator, set as

$$\operatorname{div}(A_F(x,\nabla u)) = \operatorname{div}\left(F^{p-1}(\nabla u)\nabla F(\nabla u) + a(x)F^{q-1}(\nabla u)\nabla F(\nabla u)\right),$$

with F a positively homogeneous Minkowski norm. They also assumed that $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary, λ is a real parameter, $p^* = Np/(N-p)$, exponent $\gamma \in (0,1)$, g is a subcritical term, $2 \leq p < q < N$ and (a_1) holds true.

First critical problem

- C. Farkas, A. Fiscella, P. Winkert, *Singular Finsler double phase problems with nonlinear boundary condition*, Adv. Nonlinear Stud. (2021)

We study the following problem

$$\begin{cases} -\operatorname{div}(A_F(x,\nabla u)) + u^{p-1} + a(x)u^{q-1} = u^{p^*-1} & \\ \qquad\qquad\qquad\qquad + \lambda\left(u^{\gamma-1} + g_1(x,u)\right) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ A_F(x,\nabla u)\cdot\nu = u^{p_*-1} + g_2(x,u) & \text{on } \partial\Omega, \end{cases} \qquad (\mathscr{P}_N)$$

still driven by the Finsler double phase operator

$$\operatorname{div}(A_F(x,\nabla u)) = \operatorname{div}\left(F^{p-1}(\nabla u)\nabla F(\nabla u) + a(x)F^{q-1}(\nabla u)\nabla F(\nabla u)\right).$$

We assume that $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary, λ is a real parameter, $\gamma \in (0,1)$, $p^* = Np/(N-p)$ and $p_* = (N-1)p/(N-p)$, $1 < p < q < N$ and

$(\tilde{a}_1)$ $\dfrac{Nq}{N+q-1} < p,$ *while* $a : \overline{\Omega} \to [0,\infty)$ *with* $a \in L^\infty(\overline{\Omega})$.

Here $F : \mathbb{R}^N \to [0, \infty)$ is a positive homogeneous function satisfying

(f_1) $F \in C^\infty(\mathbb{R}^N \setminus \{0\})$;

(f_2) *the Hessian matrix* $\nabla^2(F^2/2)(\xi)$ *is positive definite for all* $\xi \neq 0$;

(f_3) *the reversibility* $r_F = \max_{\xi \neq 0} \dfrac{F(-\xi)}{F(\xi)}$ *is finite.*

We point out that in general $r_F \geq 1$. When F coincides with the Euclidean norm, that is $F(\xi) = \left(\sum_{i=1}^N |\xi_i|^2\right)^{1/2}$ for $\xi \in \mathbb{R}^N$, of course $r_F = 1$.

In this direction, the operator

$$\operatorname{div}(A_F(x, \nabla u)) = \operatorname{div}\left(F^{p-1}(\nabla u)\nabla F(\nabla u) + a(x)F^{q-1}(\nabla u)\nabla F(\nabla u)\right)$$

reduces to the classical double phase operator

$$\operatorname{div}(A(x, \nabla u)) = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right).$$

First critical problem

The natural solution space for $(\mathscr{P}_N)$ is the Musielak-Orlicz Sobolev space $W^{1,\mathcal{H},F}(\Omega)$, defined by

$$W^{1,\mathcal{H},F}(\Omega) := \left\{u \in L^{\mathcal{H}}(\Omega) : \ F(\nabla u) \in L^{\mathcal{H}}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{1,\mathcal{H},F} := \|u\|_{\mathcal{H}} + \|F(\nabla u)\|_{\mathcal{H}}.$$

Here, $g_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are *Carathéodory* functions verifying

(h) $g_1(x,t) = g_2(x,t) = 0$ *for all* $t \leq 0$ *and for a.e.* $x \in \Omega$ *and* $x \in \partial\Omega$, *respectively. Furthermore, there exist* $\theta_1 \in (1,p)$, $r_1 \in [p,p^*)$, $r_2 \in (p,p_*)$ *as well as nonnegative constants* a_1, a_2 *and* b_1 *such that*

$$\begin{aligned} g_1(x,t) &\leq a_1 t^{r_1-1} + b_1 t^{\theta_1-1} && \textit{for a. e. } x \in \Omega \textit{ and for all } t \geq 0,\\ g_2(x,t) &\leq a_2 t^{r_2-1} && \textit{for a. e. } x \in \partial\Omega \textit{ and for all } t \geq 0. \end{aligned}$$

THEOREM (FARKAS & FISCELLA & WINKERT, ANS 2021)

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary, and let $\gamma \in (0,1)$. Let $1 < p < q < N$ and $a(\cdot)$ satisfy $(\widetilde{a_1})$. Let (f_1)-(f_3), (h) hold true.

Then, there exists $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$ problem

$$\begin{cases} -\operatorname{div}(A_F(x,\nabla u)) + u^{p-1} + a(x)u^{q-1} = u^{p^*-1} & \\ \qquad\qquad + \lambda\left(u^{\gamma-1} + g_1(x,u)\right) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ A_F(x,\nabla u)\cdot\nu = u^{p_*-1} + g_2(x,u) & \text{on } \partial\Omega, \end{cases} \qquad (\mathscr{P}_N)$$

admits a positive weak solution.

Clearly, the weak solutions of

$$\begin{cases} -\operatorname{div}(A_F(x,\nabla u)) + u^{p-1} + a(x)u^{q-1} = u^{p^*-1} \\ \qquad\qquad\qquad\qquad\qquad + \lambda\left(u^{\gamma-1} + g_1(x,u)\right) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ A_F(x,\nabla u)\cdot\nu = u^{p_*-1} + g_2(x,u) & \text{on } \partial\Omega, \end{cases} \qquad (\mathscr{P}_N)$$

are the critical points of the Euler-Lagrange functional $J_\lambda : W^{1,\mathcal{H},F}(\Omega) \to \mathbb{R}$, given by

$$J_\lambda(u) := \frac{1}{p}\|F(\nabla u)\|_p^p + \frac{1}{q}\|F(\nabla u)\|_{q,a}^q + \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|u\|_{q,a}^q - \frac{1}{p^*}\|u_+\|_{p^*}^{p^*}$$
$$- \frac{\lambda}{\gamma}\int_\Omega (u_+)^\gamma\,dx - \lambda\int_\Omega G_1(x,u_+)\,dx - \frac{1}{p_*}\|u_+\|_{p_*,\partial\Omega}^{p_*} - \int_{\partial\Omega} G_2(x,u_+)\,d\sigma,$$

where $u_\pm = \max(\pm u, 0)$, which is not differentiable in $W^{1,\mathcal{H},F}(\Omega)$. Also, we have to deal with the lack of compactness of embeddings

$$W^{1,\mathcal{H},F}(\Omega) \hookrightarrow L^{p^*}(\Omega), \qquad W^{1,\mathcal{H},F}(\Omega) \hookrightarrow L^{p_*}(\partial\Omega). \tag{3}$$

For this, let $\Psi : (0, \infty) \to \mathbb{R}$ set as

$$\Psi(t) := \frac{1}{p2^{p-1}{r_F}^p} - \frac{2^{p^*-1}c_{p^*}^{p^*}}{p^*}t^{p^*-p} - \frac{2^{p_*-1}c_{p_*}^{p_*}}{p_*}t^{p_*-p},$$

with $c_{p^*}, c_{p_*} > 0$ given by (3). Let $\varrho^* > 0$ be the unique value such that $\Psi(\varrho^*) = 0$.

Then, for any $\varrho \in (0, \varrho^*)$ we restrict J_λ to the closed convex set $B_\varrho(0)$, which is given by

$$B_\varrho(0) := \left\{u \in W^{1,\mathcal{H},F}(\Omega) : \ \|u\|_p + \|F(\nabla u)\|_p \leq \varrho\right\},$$

and by minimization and truncation arguments we get our solution.

- C. Farkas, A. Fiscella, P. Winkert, *On a class of critical double phase problems*, J. Math. Anal. Appl. (2022)

We study the following problem

$$\begin{cases} -\operatorname{div}(A(x,\nabla u)) = \lambda |u|^{r-2}u + |u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathscr{P}_D)$$

driven by

$$\operatorname{div}(A(x,\nabla u)) = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right),$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary, λ is a real parameter, $r \in (1,p)$, $p^* = Np/(N-p)$, $1 < p < q < N$ and

(a_1) $\dfrac{q}{p} < 1 + \dfrac{1}{N}$, *while* $a : \overline{\Omega} \to [0,\infty)$ *is Lipschitz continuous.*

THEOREM (FARKAS & FISCELLA & WINKERT, JMAA 2022)

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary and let $r \in (1, p)$. Let $1 < p < q < N$ and $a(\cdot)$ satisfy (a_1).

Then, there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ problem

$$\begin{cases} -\operatorname{div}(A(x, \nabla u)) = \lambda |u|^{r-2}u + |u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (\mathscr{P}_D)$$

admits infinitely many weak solutions with negative energy values.

Clearly, the weak solutions of

$$\begin{cases} -\operatorname{div}(A(x,\nabla u)) = \lambda |u|^{r-2}u + |u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathscr{P}_D)$$

are the critical points of the Euler-Lagrange functional $I_\lambda : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$, given by

$$I_\lambda(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_{q,a}^q - \frac{\lambda}{r}\|u\|_r^r - \frac{1}{p^*}\|u\|_{p^*}^{p^*},$$

which is of class C^1 on $W_0^{1,\mathcal{H}}(\Omega)$. However, I_λ is not bounded from below. For this, we introduce a truncation argument to control the critical term, inspired by

- J. García Azorero, I. Peral Alonso, *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term*, Trans. Amer. Math. Soc. (1991)

Furthermore, in order to study the compactness of

$$I_\lambda(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_{q,a}^q - \frac{\lambda}{r}\|u\|_r^r - \frac{1}{p^*}\|u\|_{p^*}^{p^*}$$

in the whole space $W_0^{1,\mathcal{H}}(\Omega)$, we have to deal with the lack of compactness of embedding

$$W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{p^*}(\Omega).$$

Even if $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$, the application of the Lions' concentration compactness principle fails. For this, we exploit a suitable convergence analysis of gradients, inspired by

- L. Boccardo, F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Anal. (1992)

Open question

Open question: which is the optimal exponent of a Sobolev type embedding for $W^{1,\mathcal{H}}(\Omega)$?

Open question

Open question: which is the optimal exponent of a Sobolev type embedding for $W^{1,\mathcal{H}}(\Omega)$?

We know from paper

- X. Fan, *An imbedding theorem for Musielak-Sobolev spaces*, Nonlinear Anal. (2012)

that $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{\mathcal{H}_*}(\Omega)$, where $\mathcal{H}_*$ is the Sobolev conjugate function of $\mathcal{H}$. That is, $\mathcal{H}_* : \Omega \times [0,\infty) \to [0,\infty)$ is such that

$$\mathcal{H}_*^{-1}(x,s) := \int_0^s \frac{\mathcal{H}_1^{-1}(x,\tau)}{\tau^{\frac{N+1}{N}}} d\tau, \text{ for any } (x,s) \in \Omega \times [0,\infty)$$

$$\mathcal{H}_1(x,\tau) := \begin{cases} \tau \mathcal{H}(x,1) & \text{if } 0 \le \tau < 1, \\ \mathcal{H}(x,\tau) & \text{if } \tau \ge 1. \end{cases}$$

However, we do not know how $\mathcal{H}_*$ explicitly looks like in the double phase setting.

Thank you for your attention!